Tradeoffs between quantization and packet loss in networked control of linear systems

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Abstract

In this paper, we consider to derive the coarsest memoryless quantizer which can stabilize a single-input discrete-time linear time-invariant system with stochastic packet loss in the sense of stochastic quadratic stability. We show that the upper bound of the coarseness is strictly given by the packet loss probability and the unstable poles of the plants. We furthermore deal with permissible dead-zone width around the origin of the quantizers and time-varying finite quantizers in order to realize control using finite quantization steps.

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1. Introduction

In recent years, networked control systems have been actively investigated in the field of control theory and one of the interests is to find the relationship between the permissible coarseness of transmitted signals for stabilization and the properties of plants. Some of the recent works on this topic include (Brockett & Liberzon, 2000; Elia & Mitter, 2001; Fu & Xie, 2005; Goodwin, Haimovich, Quevedo, & Welsh, 2004; Nair & Evans, 2004; Tatikonda & Mitter, 2004a; Tsumura & Maciejowski, 2003; Wong & Brockett, 1999). In particular, in Elia and Mitter (2001) a stabilization problem via quantized input signals is considered and the coarsest memoryless quantizer for stabilization of single-input discrete-time linear time-invariant systems is derived. A notable point is that the upper bound of the coarseness is given only by the unstable poles of the plants. This result is also extended to LQR type problems (Fu & Xie, 2005) and adaptive control problems (Hayakawa, Ishii, & Tsumura, 2009a,b).

Another problem we should deal with for networked control systems is the packet loss in data transmission. This problem arises when unreliable communication channels are used such as wireless networks or general-purpose channels. Clearly, losses of signals cause performance degradation or can make a closed-loop system unstable. Some research groups have dealt with this problem. LQ type control problems are considered in Imer, Yüksel, and Başar (2006), and H∞ control approaches were proposed in Sellier and Sengupta (2005) and Ishii (2008a). Sinopoli et al. (2004), studied stabilization in state estimation problems under packet losses. In Elia (2005) and Ishii (2008b), the mean square stability of feedback control systems is investigated and the upper limit of loss probability is given in terms of the unstable poles of the plants. For the scalar case, this was shown in Hadjicostis and Touri (2002).

In spite of the above significant results showing the relationships between “the unstable poles of plants and the coarseness of quantization (Elia & Mitter, 2001)” and between “the unstable poles of plants and the packet loss probability (Elia, 2005; Ishii, 2008b),” in real communication channels, it is more realistic to assume that the channel contains both quantization and stochastic packet losses. A natural extension of our interests is on the relationship among the three properties above for such networked control systems.
The system is an independent and where

\[
\alpha < \alpha_{\text{sup}} = \frac{1}{\prod_{i} |\lambda_{+}^{i}|^{\frac{2}{p}}}. 
\]

In this paper, we consider the following discrete-time linear system:

\[
G : x(k + 1) = Ax(k) + B\hat{v}(k),
\]
where \(x(k) \in \mathbb{R}^{n}\) is the state vector, \(\hat{v}(k) \in \mathbb{R}\) is the control input, \(A \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times 1}\). Assume that \((A, B)\) is stabilizable and \(A\) is unstable.

We explain how the control signal is processed when it is transmitted from a controller to the control input of \(G\) according to Fig. 1. At first, the control signal \(u(k)\) from the controller is quantized at the controller side before it is sent over a communication channel. The quantization is given by

\[
u(k) = q(u(k)),
\]
where \(q(\cdot)\) is a memoryless quantizer and \(u(k) \in \mathbb{R}\) is an ordinary analog control input generated by a static state feedback controller \(K(\cdot)\).

In addition, we assume that packet losses occur with probability \(\alpha\) at the input-side channel of the plant. In this paper, we employ a simple scheme where the packet loss sets \(\hat{v}(k) = 0\), and hence the system can be described as

\[
x(k + 1) = Ax(k) + B\hat{v}(k)\nu(k),\]
where \(\hat{v}(k)\) is a 0–1 random variable with a probability distribution given by

\[
Pr(\hat{v}(k) = i) = \begin{cases} \alpha, & i = 0, \\ 1 - \alpha, & i = 1, \\ 0 \leq \alpha < 1. \end{cases}
\]

The reason why we deal with the case that the quantization is limited to the plant input side is that it is one of the basic setups. It is also a model where a large difference exists in the capacities for transmissions to and from the controller such as in a wireless-networked control system or a large-scale plant.

We next describe the stability we employ in this section. Consider the following discrete-time system:

\[
x(k + 1) = f(x(k), \theta(k)),
\]
where \(x(k) \in \mathbb{R}^{n}\) is the state, and \(\theta(k) \in \{0, \ldots, N - 1\}\) represents the mode of the system. The mode is an independent and identically distributed stochastic process with probabilities \(\alpha_{i} = Pr(\theta(k) = i)\). The function \(f(x, \theta)\) satisfies \(f(0, \theta) = 0\) for arbitrary \(\theta\). Thus, the origin \(x = 0\) of the system is an equilibrium point.

For this system we define the following stability:

**Definition 2.1.** For the system (4), the equilibrium point at the origin is stochastically quadratically stable if there exists a positive-definite function \(V(x) = x^{T}P\) and a positive-definite matrix \(R\) such that

\[
\Delta V = E[V(x(k + 1)) - V(x(k))] 
\leq -x(k)^{T}Rx(k), \quad \forall x(k) \in \mathbb{R}^{n}.
\]

**Remark 2.1.** The condition (5) is sufficient for the origin of the system (4) to be mean square stable (see, e.g., Ji and Chizeck (1990)), i.e., for every initial state \(x_{0}\),

\[
\lim_{k \to \infty} E[\|x(k)\|^{2}|x_{0}] = 0.
\]

The important point on the condition (5) is that the absolute averaged decreasing rate of a Lyapunov function \(V\) is larger than or equal to a quadratic form of \(x\). Also we should note that another condition \(\Delta V < 0, \forall x\), does not necessarily guarantee stability for the “stochastic” nonlinear system different from the case Elia and Mitter (2001). The matrix \(R(>0)\) in (5) regulates the convergence rate of \(x\) and it is critical for the moment of \(x\) as shown in Proposition 3.1 and Theorem 3.1, Section 3, where we deal with a case that the quantizer has a dead-zone.

In this section, our objective is to find the coarsest quantizer \(q(\cdot)\) which achieves stochastic quadratic stability for the system (3). The coarseness of a quantizer \(q(\cdot)\) is defined as (Elia & Mitter, 2001)

\[
d = \lim_{\epsilon \to 0} \sup_{\epsilon \to 0} \frac{\mathbb{E}[u(\epsilon)]}{-\ln \epsilon}.
\]
where \(\mathbb{E}[u(\epsilon)]\) denotes the number of levels that the quantizer \(q(\cdot)\) has in the interval \([\epsilon, 1/\epsilon]\).

Elia and Mitter (2001) showed that the coarsest quantizer for the quadratic stabilization in the case of no packet loss is logarithmic and the coarsest expansion ratio \(\rho_{\text{sup}}\) (which is strictly defined later) is given by

\[
\rho_{\text{sup}} = \prod_{i} |\lambda_{+}^{i}|^{\frac{2}{p}}.
\]

where \(\lambda_{+}^{i}\) represents the unstable poles of the plant. On the other hand, in Elia (2005) and Ishii (2008b), a necessary and sufficient condition on \(\alpha\) for the mean square stabilizability in the case of no quantization is given as

\[
\alpha < \alpha_{\text{sup}} = \frac{1}{\prod_{i} |\lambda_{+}^{i}|^{\frac{2}{p}}}. 
\]

In this paper, we consider the effects of both quantization and packet losses and the natural extension of our interests is “What relationship between \(\rho_{\text{sup}}, \alpha\) and \(\lambda_{+}^{i}\) does there exist?” We provide a complete answer to this question in the following theorem, which unifies the results (8) and (9).

**Theorem 2.1.** The coarsest quantizer \(q_{c}(\cdot)\) with which the system (3) is stochastically quadratically stable is given as:

\[
q_{c}(u) = \begin{cases} v, \quad u \in \left[0, \frac{\rho_{\text{sup}} + 1}{2\rho_{\text{sup}}} v \right], \\ v^{-1}, \quad u \in \left[-\frac{\rho_{\text{sup}} + 1}{2\rho_{\text{sup}}} v, 0\right], \\ 0, \quad u = 0, \end{cases}
\]

Fig. 1. Stabilization via quantized signals with stochastic packet losses.

![Diagram](image-url)
In general, logarithmic quantizers are given by
\[ \text{some part of the proof of } U \text{atively.} \]

The curved surface represents are found on the i.e., high packet loss probabilities require high resolution quantiz-

special cases. The result also shows a tradeoff between α and ρ, i.e., high packet loss probabilities require high resolution quantizers and vice versa for stability.

Proof. The outline of the proof is similar to that of Elia and Mitter (2001). We focus on the difference in the following, but also refer to some parts of the proof of Elia and Mitter (2001) in order to make the proof in a self-contained form.

At first, in order to find the coarsest quantization from u to v, we write as \( v = u \) in form, and find \( P \) such that there exists an input \( u \) satisfying \( \Delta V(x) < 0 \) for a given \( x \) and a set \( U(x, P, \alpha) \) of \( u \) for such \( P \). By substituting (3) into the left-hand side of (5), we get

\[
\Delta V = E[V(Ax + \theta Bu)|x] - V(x) = x^\top(A^\top PA - P)x + 2(1 - \alpha)x^\top A^\top PBu + (1 - \alpha)B^\top PBU^2 < 0.
\]

Then, \( U(x, P, \alpha) \) can be given by

\[
U(x, P, \alpha) = \{ u \in \mathbb{R} | \Delta V < 0 \} = \{ u \in \mathbb{R} | u_ - < u < u_ + \}
\]

and assume \( P \) satisfies the inequality (13).

According to Elia and Mitter (2001), \( x \) can be parametrized by

\[
x = Q(P, \alpha)^{-1}A^\top PBu_1 + Q(P, \alpha)^{-1/2}p_2,
\]

where \( p_1 \in \mathbb{R} \) and \( p_2 \in \mathbb{R}^n \) with \( p_2 \perp Q^{-1/2}A^\top PB \), and therefore, \( u_- \) and \( u_+ \) can be written as

\[
u_\pm = \frac{B^\top PAQ(P, \alpha)^{-1}A^\top PB}{B^\top P} \pm \frac{\sqrt{B^\top PAQ(P, \alpha)^{-1}A^\top PB} + \|p_2\|^2}{B^\top P}.
\]

Then, for a given \( p_1 \), the allowable width \( |u_+ - u_-| \) becomes minimum when \( p_2 = 0 \). Following Elia and Mitter (2001), we consider the case where \( x \) is in such a worst case 1-dimensional subspace \( u | p_2(x) = 0 \), and then we get

\[
u_\pm = \tilde{p}_1(\gamma \pm 1), \quad \tilde{\gamma} = \frac{B^\top PAQ(P, \alpha)^{-1}A^\top PB}{B^\top P}.
\]

That is, the allowable inputs become a convex sector region where the origin is the vertex in the \( p_1 \)-\( u \) plane.

Here, we should take care of the relationship between the stochastic quadratic stability (5) and \( u_\pm \) as follows. The quantities \( u_\pm \) are the boundaries of \( u \) satisfying \( \Delta V < 0 \). Therefore, for the actual quantizer in order to attain (5), we should introduce \( \gamma > \tilde{\gamma} \) and redefine both ends of the sector for \( u \) as \( u_\pm = \tilde{p}_1(\gamma \pm 1) \) for the existence of \( R > 0 \) in (5). However, for any \( u \) in \( u_- \leq u \leq u_+ \), with \( \gamma > \tilde{\gamma} \), there always exists \( R \) such that \( \Delta V \leq -\xi^\top R \xi \forall u \) from the definition of the coarseness (7). In the sense above, \( u_\pm \) given in (15) are also the upper and lower bounds of possible \( u \) for a given \( x \) (i.e., \( \tilde{p}_1 \)) in order to satisfy (5) and \( \gamma \) is the lower bound of the possible \( \gamma \). Hereafter, we focus on these boundaries.

According to Elia and Mitter (2001), the coarsest quantizer of \( u \) is the following coarsest piecewise constant function in the sector in the \( p_1 \)-\( u \) plane explained above:

\[
q(u) = \begin{cases} v_+, & u \in \left(0, \frac{\rho + 1}{2}\right), \\ v_-, & u \in \left[\frac{\rho + 1}{2}, \frac{\rho + 1}{2}\right), \\ 0, & u = 0, \end{cases}
\]

where \( v_0 > 0, v_1 = \rho v_0, i \in \mathbb{Z}, \) and \( (P, \alpha) = \frac{\rho + 1}{\rho - 1} \). Moreover, the quantization density \( d \) is given as \( d = \frac{2}{\rho + 1} \). Note that \( d \) is monotonically decreasing in \( \rho \), and \( \rho = \frac{\rho + 1}{\rho - 1} \) is also monotonically decreasing in \( \gamma > 1 \), and then, the minimization of \( d \) reduces to that of \( \tilde{\gamma} \) as follows:

\[
\gamma_{\text{ad}} = \inf_{P > 0, \omega_0, \omega_1} \tilde{\gamma}.
\]

The above problem is equivalent to the minimization of \( \gamma \) subject to

\[
A^\top PA - P - \left(1 - \frac{1 + \alpha(\gamma^2 - 1)}{\gamma^2}\right) \times A^\top PB \left(\frac{B^\top P}{B^\top P} - \frac{1}{B^\top P}\right) < 0, \quad P > 0,
\]

from (13) and (15).

From Lemma A.1 of Ishii (2008b), the condition for the existence of \( P \) and \( \gamma \) satisfying (17) is equivalent to the existence of a feedback gain \( F \) and \( \gamma \) such that

\[
\|F(zI - A - BF)^{-1}B\|_\infty < \frac{\gamma}{\sqrt{1 + \alpha(\gamma^2 - 1)}}
\]
in a state feedback system. Note that the right-hand side above is monotonically increasing in $\gamma > 1$. Then, the lower bound of $\gamma$ can be given by the minimization of the left-hand side. By employing Lemma 2.4 in Fu and Xie (2005), which gives the lower bound of $H_{\infty}$ norm of the state feedback systems, we get $\nu_{id} = \sqrt{\frac{1-\alpha}{\gamma - 1}}$.

**Remark 2.3.** We remark that the proof above is first based on the condition $\Delta V < 0$. This is to obtain the critical bound $\rho_{sup}$ on the expansion ratio $\rho$ for the quantizer. However, as discussed in Remark 2.1, to guarantee stochastic quadratic stability when a quantizer with $\rho$ smaller than $\rho_{sup}$ is used, we must check the existence of the matrix $R > 0$ in (5). In the proof, we show that this holds true.

3. Dead-zone width design of quantizer for practical stabilization

The quantizer of the form (10) requires infinitesimal quantization levels around the origin. In order to keep the complexity of the signals sent over the channels reasonable, the quantizer should have a dead-zone around the origin though in such a case, asymptotic stability as (6) cannot be guaranteed. On the other hand, from the result in the previous section, the logarithmic quantizer is appropriate except for the neighborhood at origin. Thus, in this section, we consider practical stabilization of control systems using a logarithmic quantizer with dead-zone under packet losses.

The problem setting in this section follows that in the previous section and is as follows: The plant is (3) with stochastic packet losses where the loss probability $\alpha$ is assumed to satisfy (9). In this section, the quantizer $q_{dead}(\cdot)$ is assumed to be memoryless and logarithmic with a dead-zone around origin and it can be expressed as

$$q_{dead}(u) = \begin{cases} v_i, & u \in \left(\frac{\rho + 1}{2\rho} r_i, \frac{\rho + 1}{2rho} v_i\right), \\ -v_i, & u \in \left(-\frac{\rho + 1}{2\rho} v_i, -\frac{\rho + 1}{2\rho} r_i\right), \\ 0, & u \in [-v, v], \end{cases}$$

(18)

$v_i = \rho^{i} v_0$, $v_0 > 0$, $i \in \mathbb{Z} \cap [0, \infty)$,

where the coarseness $\rho$ of the quantizer satisfies $1 < \rho < \rho_{sup}$. Note that the relationship between $v$ and $v_0$ is $v = \frac{\rho^{i+1}}{\rho^{i}} v_0$.

Moreover, $u(k)$ is given by $u(k) = Kx(k) = \frac{B_P}{B_P}$, where $P \in \mathbb{R}^{n \times n}$ is the positive-definite matrix satisfying a Ricatti type inequality:

$$A^T PA - P - (1 - \alpha) \left(1 - \frac{\rho - 1}{\rho + 1} \right)^2 \times A^T PB (B^T PB)^{-1} B^T PA < 0,$$

(19)

which is given by (17) and $\gamma$ satisfying (17) with the relationship $\rho = \frac{\gamma^{-1}}{\gamma + 1}$.

Then, we further define a positive-definite matrix $R$ and a positive number $\delta$ such as

$$R = P - A^T PA + (1 - \alpha) \left(1 - \frac{\rho - 1}{\rho + 1} \right)^2 \times A^T PB (B^T PB)^{-1} B^T PA - \delta I,$$

(20)

where $\delta$ is chosen so that $R$ is positive definite.

**Remark 3.1.** When $v \rightarrow 0$, the quantizer above goes to (16) in the previous session. Then the origin of the system (3) is stabilized in the sense of stochastic quadratic stability (5).

We next define the stability considered in this section.

**Definition 3.1.** For the system (4), the equilibrium point at the origin is mean square practically stable for a given $\epsilon > 0$ if for every initial state $x_0$,

$$\lim_{k \rightarrow \infty} E[\|x(k)\|^2|x_0] \leq \epsilon.$$

(21)

We should note that the quantizers are introduced to decrease the amount of information which is transferred over the channels. Therefore, the focus of the problem here is to find how large the dead-zone can be taken for a given stability specification $\epsilon$.

**Remark 3.2.** In Elia and Mitter (2001), a similar problem is considered for deterministic systems without packet losses. In such a case, $R$ in (5) does not affect the relationship between $\epsilon$ and the width $\nu$ of the dead-zone. On the other hand, in the stochastic systems considered in this paper, $R$ plays an important role as we will explain later.

We now present the result as the following theorem:

**Theorem 3.1.** Under the control law $v(k) = q_{dead}(Kx(k))$ with the quantizer in (18), the origin of the system (3) is mean square practically stable for a given $\epsilon > 0$ if

$$v \leq \sqrt{\frac{c}{\beta}},$$

where $c > 0$ is given as

$$c = \frac{\lambda_{min}(P)}{\lambda_{max}(R)} \epsilon,$$

and $\beta$ is the minimum positive number such that

$$\beta \Phi - \Pi - \tau_1 (\tilde{P}_A - \tilde{P} + \tilde{R}) \geq 0, \quad \exists \tau_1 \geq 0,$$

$$\beta \Phi - \Pi - \tau_2 (\tilde{P}_A - \tilde{P} + \tilde{R}) \geq 0, \quad \exists \tau_2 \geq 0,$$

$$\Phi = \text{diag} \{0 \ldots 0 1\}, \quad \tilde{P} = T^T PT, \quad \tilde{R} = T^T RT,$$

$$\tilde{P}_A = \tilde{A}^T PA, \quad \tilde{A} = T^{-1} AT.$$

Moreover, the matrix $T \in \mathbb{R}^{n \times n}$ is an invertible matrix defined by

$$T = \begin{bmatrix} W & K^T \\ \|K\|^2 & 0 \end{bmatrix},$$

(24)

and $W \in \mathbb{R}^{n \times (n-1)}$ is an arbitrary matrix satisfying $KW = 0$.

**Remark 3.3.** The inequalities (22) and (23) are LMIs of $\tau_1$, $\tau_2$, and $\beta$ and are thus convex. Therefore, the minimum $\beta$ satisfying the conditions can be computed by efficient algorithms.

For the proof of this theorem, we first show the following lemma and proposition.

**Lemma 3.1.** For the system (3), the quantizer (18) and a given positive number $c$, the following are satisfied:

$$E[|x(k+1)|^2|x(k)|] \leq c, \quad \forall x(k) \in \mathcal{L}_c(c),$$

(25)

$$\Delta V = E[V(x(k+1))|x(k)] - V(x(k)) \leq -\tau_1 \|R\| x(k), \quad \forall x(k) \in \mathbb{R}^n \setminus \mathcal{L}_c(c),$$

(26)

where $\mathcal{L}_c(c)$ is the following level set: $\mathcal{L}_c(c) = \{x \in \mathbb{R}^n | V(x) \leq c\}$, $v$ in the quantizer (18) is a positive number satisfying $v \leq \sqrt{\frac{c}{\beta}}$, and $\beta$ is a solution of (22) and (23).

**Proof.** At first, consider the following transformation with $T$ of (24): $x = T [r_1^T, r_2^T]^T$, where $r_1 \in \mathbb{R}^{n-1}, r_2 \in \mathbb{R}$. Note that $r_2 = u$ from the definition of $T$. 

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Next define $\mathcal{D} := \{ x \in \mathbb{R}^n \mid -v \leq x \leq v \}$. This represents the region in the state space such that the output of the quantizer becomes $0$. The region $\mathcal{D}$ can be divided into the following two parts:

$\mathcal{D}_\gamma := \{ x \in \mathcal{D} \mid x^T(A^T P A - P + R)x > 0 \}$, 
$\mathcal{D}_\Omega := \{ x \in \mathcal{D} \mid x^T(A^T P A - P + R)x \leq 0 \}$.

Here the important region is $\mathcal{D}_\gamma$ in order to show (25) and (26) are satisfied. The reason is the following. In the dead-zone free case, the quantization (18) for $v = 0$ satisfies $\Delta V \leq -x^T Rx$ for the whole space of $x$, and therefore, (26) is satisfied for $x(k) \in \mathbb{R}^n \setminus \mathcal{L}_V(c)$.

From the above consideration, the problem reduces to finding a condition on $v$ to guarantee $x^T P x \leq c, x^T A^T P A x \leq c, \forall x \in \mathcal{D}$. Instead of $x$, we represent the above by $r_1$ and $r_2$ as

$$\begin{cases} r_1^T r_2 P \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \leq c, & r_1^T r_2 \hat{P}_A \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \leq c. \\ \forall r_1, r_2 \text{ such that} & \begin{bmatrix} r_1^T \\ r_2^T \end{bmatrix} \begin{bmatrix} \beta P - \delta \hat{P} + \delta \hat{R} \\ \beta \hat{P} - \beta \hat{A} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} > 0, \quad -v \leq r_2 \leq v. \end{cases}$$

(27)

From the convexity of the condition (27), it is enough to evaluate them at $r_2 = v$.

Now we consider the following variable transformation from $c$ and $r_1$ to $\beta$ and $r_1$ such as $c = v^2 \beta^2 (\beta > 0)$ and $r_1 = v r_1$. We get the following by the definition of $\Phi$:

$$\begin{cases} r_1^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} \beta \Phi - \delta \hat{P} + \delta \hat{R} \\ \beta \hat{P} - \beta \hat{A} \end{bmatrix} \begin{bmatrix} r_1 \end{bmatrix} > 0, \\ \forall r_1 \text{ such that} & r_1^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} \beta \Phi - \delta \hat{P} + \delta \hat{R} \\ \beta \hat{P} - \beta \hat{A} \end{bmatrix} \begin{bmatrix} r_1 \end{bmatrix} > 0. \end{cases}$$

By the $S$-procedure, this condition is equivalent to the existence of non-negative $r_1$ and $r_2$ satisfying the following:

$$\begin{cases} \beta \Phi - \delta \hat{P} - \tau_1 \beta \hat{A} - \delta \hat{R} > 0, \\ \beta \Phi - \delta \hat{P} - \tau_1 \beta \hat{A} - \delta \hat{R} > 0. \end{cases}$$

Therefore, for $v$ satisfying $v \leq \sqrt{\frac{2}{\beta}}$, (25) and (26) are satisfied. $

**Proposition 3.1.** For the system (3) with given positive-definite matrices $P$ and $R$, and a given number $c > 0$, suppose (25) and (26) are guaranteed. Then, for arbitrary $x_0$, the following holds:

$$\lim_{k \to \infty} \mathbb{E} [\|x(k)\|^2 | x_0] \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} c. \quad (28)$$

**Proof.** With (26), in the case of $x(k) \notin \mathcal{L}_V(c)$, we have

$$\mathbb{E} [\frac{\|x(k+1)\|^2}{\|x(k)\|^2}] = 1 - \frac{x(k)^T (\hat{R} x(k))^{(1)}}{x(k)^T P x(k)} \leq a, \quad a := 1 - \frac{\lambda_{\min}(R)}{\lambda_{\max}(P)}$$

(0 < a < 1). Then, we get

$$\mathbb{E} [\frac{\|x(k+1)\|^2}{\|x(k)\|^2}] = 1 - \frac{x(k)^T (\hat{R} x(k))^{(1)}}{x(k)^T P x(k)} \leq a, \quad (X(k)) \in \mathbb{R}^n \setminus \mathcal{L}_V(c).$$

On the other hand, in the case of $x(k) \in \mathcal{L}_V(c)$, (25) holds. By unifying these inequalities, we obtain

$$\mathbb{E} [\|x(k+1)\|^2] \leq \mathbb{E} [\|x(k)\|^2] + c, \quad \forall x(k) \in \mathbb{R}^n.$$
One assumption we make in this section is that the coder in Fig. 1 knows the value of \( \theta(k) \) at time \( k+1 \). This may be realized by an acknowledgement message sent from the decoder if, e.g., more power is available there for transmission (e.g., Liner et al. (2006) and Ishii (2008)). Another approach is to add an observer-like system on the sensor side to estimate the input applied (Sahai & Mitter, 2006). If this assumption fails to hold, a more conservative design would be necessary.

The stability definition used in this section is as follows.

**Definition 4.1.** The closed-loop system under the control law in (30) is said to be stable with probability 1 (w.p. 1) if \( \|x(k)\| \to 0 \) as \( k \to \infty \) w.p. 1.

We provide a preliminary result. At first, take \( N \) large enough that
\[
N \geq \log_2 \frac{F_0}{1 - \frac{\min\{P, \lambda\}}{\min\{P, \lambda\}}}
\]
where
\[
F_0 = \sqrt{\frac{\sqrt{2}(\rho + 1)\eta_+}{2\sqrt{\min\{P\}}\|Q^{-1}AP\|}},
\]
and \( \beta \) is the value given in Theorem 3.1. Let
\[
c_1(v_0) := \left( \frac{p + 1}{2\rho} \right)^2 \beta,
\]
\[
c_2(v_0) := \frac{\min\{R\}}{\min\{P\}} \lambda_{\min}\left[ \frac{(\tilde{Q}^{-1}AP\tilde{P}B)v_{n-1}}{\eta_+} \right]^2,
\]
and \( \beta \) is the value given in Theorem 3.1. Let
\[
\tilde{Q} := \left( \frac{p - 1}{\rho + 1} \right)^2 \frac{A^TB\tilde{PB}B}{d^2} + \frac{\delta}{1 - \alpha},
\]
and \( \eta_+ := \frac{B^TPA\tilde{Q}^{-1}AP}{B^TBP} \).

A result similar to Lemma 3.1 holds as shown in the next lemma.

**Lemma 4.1.** Given \( v_0 \) > 0, under the control law \( v(k) = q_{\theta}(Kx(k)) \), the following hold:
\[
E[V(x(k+1))|x(k)] \leq c_1(v_0),
\]
\[
\forall x(k) \in L_2(c_1(v_0)),
\]
and the control input \( \tilde{v}(k) = 0 \).

Now, the control protocol for the time-varying quantization scheme of (30) is as follows: Given a positive scalar \( R_0 \) > 0, suppose the initial state satisfies \( \|x(0)\| \leq R_0 \). There are two auxiliary time-varying parameters \( v_0(k) \) and \( w(k) \) used in the scheme. Let the initial value of \( v_0 \) be
\[
v_0(0) = \sqrt{\frac{\min\{R\}}{\min\{P\}} \frac{|\eta_+|}{\|Q^{-1}AP\|}} / \beta^{\frac{N-1}{2}},
\]
and let
\[
w(k) = 1, \text{ if } x(k) \in L_2(c_1(v_0(k))),
\]
\[
0, \text{ if } x(k) \not\in L_2(c_1(v_0(k))).
\]

What need to be transmitted over the channel at each time \( k \) are the index of the quantized signal \( v(k) \) and the binary signal \( w(k) \). In both the coder and the decoder, \( v_0(k) \) can be constructed because \( \theta(k) \) is available on both sides.

The next theorem is the main result of the section and characterizes the stabilization of the system in (3).

**Theorem 4.1.** If \( \rho \in (1, \rho_{\text{sup}}) \) and \( \alpha \in (0, \alpha_{\text{sup}}) \) are sufficiently small that
\[
\eta_+ := \frac{B^TPA\tilde{Q}^{-1}AP}{B^TBP} \leq \beta^{\frac{N-1}{2}},
\]
then the closed-loop system under the control law specified above is stable with probability 1.

**Proof.** We first claim that \( x(k) \in L_2(c_2(v_0(k))) \), \( \forall k \). We show this by induction. By the assumption on \( x(0) \) and the choice of \( v_0(0) \), we have \( x(k) \in L_2(c_2(v_0(k))) \).

Now, assume \( x(k) \in L_2(c_2(v_0(k))) \). Suppose \( \theta(k) = 0 \). Then, the control input is \( \tilde{v}(k) = 0 \). Hence,
\[
V(x(k+1)) = x^T(k)A^TPx(k) \leq c_2(v_0(k) + 1).
\]
That is, \( x(k+1) \in L_2(c_2(v_0(k) + 1)) \).

Next, if \( \theta(k) = 1 \), there are two cases. The first case is when \( x(k) \not\in L_2(c_1(v_0(k))) \). We can show by (32) in Lemma 4.1 that
\[
V(x(k+1)) \leq \frac{1}{1 - \alpha} x^T(k) [\alpha A^TPA - P + R] x(k) \leq c_2(v_0(k) + 1).
\]
The second case is when \( x(k) \in L_2(c_1(v_0(k))) \). Then, by definition, \( v_0(k+1) = F_0\rho^{-1}v_0(k) \), which implies \( c_2(v_0(k+1)) = c_2(v_0(k)) \).

By construction of the quantizer, it is clear that if \( v(k) \neq 0 \), then \( V(x(k+1)) \leq V(x(k)) \leq c_2(v_0(k)) \). If \( v(k) = 0 \), then applying Lemma 4.1 to this case yields \( E[V(x(k+1))|x(k)] = V(x(k)) \leq c_2(v_0(k)) \). Hence, the claim is proved.

Now, the closed-loop system is stable when \( v_0(k) \to 0 \) as \( k \to \infty \) w.p. 1. In doing so, let \( \tilde{v}(k+1) = \psi_{\theta}(\tilde{v}(k)) \) with \( \tilde{v}(0) = v_0(0) \), where \( \psi_{\theta}(\tilde{v}(k)) = \frac{\max\{\max\{PA\}, \max\{P\}\}^{1/2}}{\max\{\max\{PA\}, \max\{P\}\}^{1/2}} \psi_{\eta}(\tilde{v}(k)) \). Due to the size of \( N \), it follows that \( F_0\rho^{-N} \leq \psi_{\eta}(\tilde{v}(k)) \) for all \( k \).

Hence, we have \( \tilde{v}(k) = \prod_{i=0}^{k-1} \psi_{\eta}(\tilde{v}(k)) \). In what follows, we show that \( \prod_{i=0}^{k-1} \psi_{\eta}(\tilde{v}(k)) \to 0 \) as \( k \to \infty \) w.p. 1. By the assumption
in (34), we have $E[\log_2 \psi_{\text{sth}}] = \alpha \log_2 \psi_0 + (1 - \alpha) \log_2 \psi_1 < 0 \iff \frac{1}{k} \sum_{i=0}^{k-1} \log_2 \psi_{\text{sth}} \rightarrow E[\log_2 \psi_{\text{sth}}] < 0, \quad k \rightarrow \infty, \ w.p. 1.

This in turn yields (Tatikonda & Mitter, 2004b)

$$\prod_{i=0}^{k-1} \psi_{\text{sth}} = 2^{\frac{1}{k} \sum_{i=0}^{k-1} \log_2 \psi_{\text{sth}}} \rightarrow 0, \quad k \rightarrow \infty, \ w.p. 1.$$ 

Thus, stability is now shown.  

The quantized control scheme in this section has a simple structure. Especially, the decoder needs to calculate only $v_0(k)$, which is a scalar. This is in contrast to the time-varying, finite data rate control approach proposed in Tatikonda and Mitter (2004b). We however note that the parameters $\rho$ and $\alpha$ satisfying the condition (34) can be conservative.

5. Numerical example

In this section, we present a numerical example to demonstrate the utility of the proposed quantizer design.

As the system (3), we considered the second-order system given by

$$x(k + 1) = \begin{bmatrix} 0 & 1 \\ 1.8 & -0.3 \end{bmatrix} x(k) + \theta(k) \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(k).$$

The system is unstable and has two poles 1.2 and -1.5.

In Fig. 2, this system corresponds to the cross section obtained by cutting the surface at $\prod |\lambda_i^{(k)}| = 1.8$. This cross section is depicted in Fig. 3. It is obvious that we need to select a communication channel with a packet loss rate $\alpha < \alpha_{\text{sup}} \simeq 0.31$. Here, we chose a channel with $\alpha = 0.25$. This in turn requires that the quantizer has its parameter as $\rho < \rho_{\text{sup}} \simeq 1.78$; we took $\rho = \rho_{\text{sup}} \times 0.95 \simeq 1.69$.

In the setting above, there exists a $P > 0$ which satisfies (19). We can then obtain $K > 0$ such that (20) holds with some $\delta$. So, we found the matrices $P$ and $R$ as

$$P = \begin{bmatrix} 3.264 & -1.180 \\ -1.180 & 3.313 \end{bmatrix}, \quad R = \begin{bmatrix} 0.043 & -0.010 \\ -0.010 & 0.033 \end{bmatrix}.$$ 

with $\delta = 0.01$, and the state feedback $K = [-1.800 \ 0.656]$. It then follows that with the quantizer $q(\cdot)$ without a dead-zone, the closed-loop system is mean square stable.

Next, we focused on stabilizing the system in the sense of (21). Specifically, we fixed $\epsilon = 200$. From Theorem 3.1, we need to use a dead-zone width of $\nu \leq \sqrt{\frac{\lambda}{\beta}} =: \nu_{\text{max}} \simeq 0.065$. With $\nu = 0.062 = \nu_{\text{max}} \times 0.95$ and the random initial condition $x(0)$ satisfying $\|x_0\|_2 = \sqrt{2} \times 100$, we computed the time responses under packet losses $10^4$ times. For comparison, we also performed a quantizer design without consideration of packet losses based on Elia and Mitter (2001), which gives a condition $\rho < 3.50 =: \rho_{\text{sup}}$. Then, we set a quantizer having $\rho = \rho_{\text{sup}} \times 0.95 \simeq 3.33$ and similarly simulated the time responses $10^4$ times with the same initial conditions and packet loss sequences.

Figs. 4 and 5 show the averages of $\|x(k)\|$ and $V(k)$ of $10^4$ samples, respectively. From them, we can see that the closed-loop system with $\rho = 1.69$ taking account of the packet losses goes very close to zero on average while the system designed by the conventional method with $\rho = 3.33$ does not and in fact the average of $V(k)$ diverges in this case.

Figs. 6 and 7 show two typical sample paths of the trajectories of $x_1(k)$ with $\rho = 1.69$ and $\rho = 3.33$. In Fig. 6, we observe that the closed-loop system with $\rho = 1.69$ converges close to zero and the system with $\rho = 3.33$ diverges, though the probability of such a case is not necessarily high. On the other hand, in Fig. 7, we present sample paths of usual cases where both trajectories converge. In this case, however, we can still observe that the convergence rate with $\rho = 1.69$ is much better than that with $\rho = 3.33$ and this fact shows the importance of $\rho$ and the influence of packet losses.

6. Conclusion

In this paper, we have considered the stabilization problem of a linear system via quantized feedback with stochastic packet losses.
We have presented the coarsest quantizer which can achieve stochastic quadratic stability. In particular, we have shown that the coarseness of the coarsest quantizer is strictly given by the packet loss probability and the unstable poles of the plant. Moreover, we have clarified the tradeoff between the level of quantization and the packet loss probability. In order to make the scheme more practical, we have also studied the cases with a dead-zone and with time-varying finite quantizers.

References


