The Second Law of Controlled Linear Stochastic Thermodynamic Systems over a Noiseless Digital Channel

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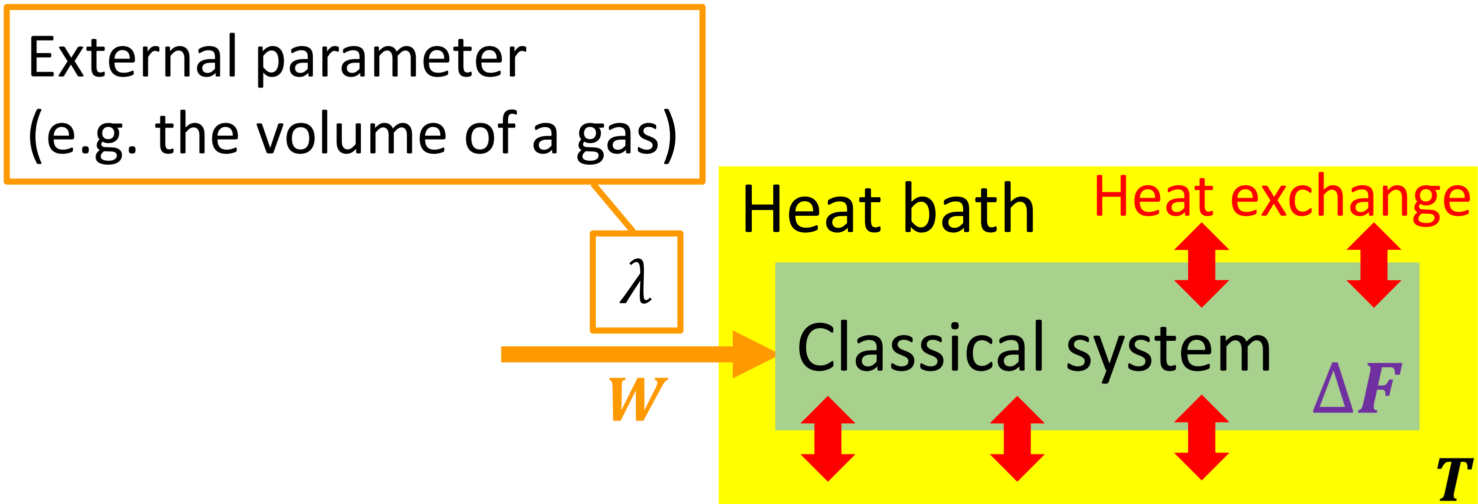
Background and Research Purpose

The Second Law of Thermodynamics for a Classical System



The work done on the system

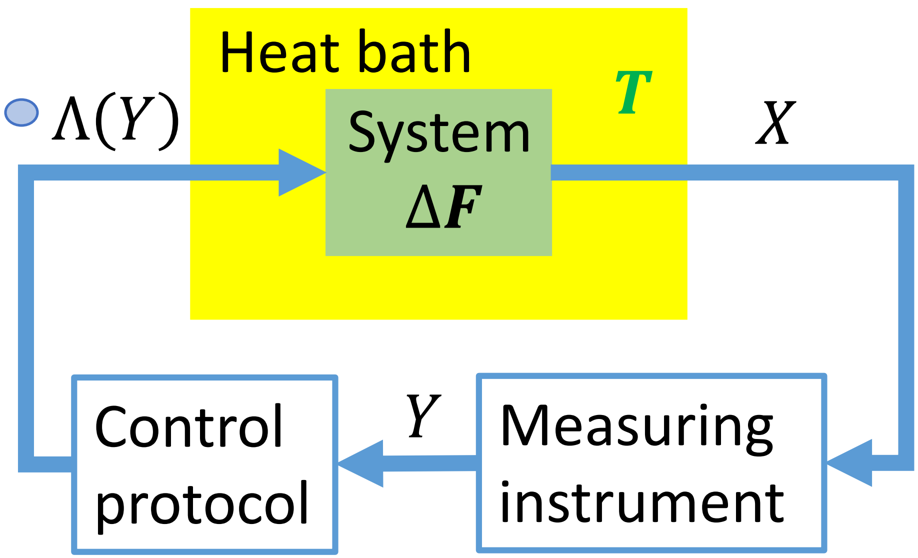
Free energy difference

$$\langle W \rangle \geq \Delta F$$

$\langle \cdot \rangle$: Ensemble average (W fluctuates due to the heat exchange)

The Second Law with Measurements and Feedback
(⇔ Maxwell's demon)

Single measurement & feedback (Sagawa & Ueda, 2010)



X: microscopic state
Y: measurement
$\Lambda(Y)$: control protocol

$$\langle W \rangle \geq \Delta F - k_B T I$$

Boltzmann constant

Mutual information between X and Y

Temperature of the heat bath

The Second Law with Measurements and Feedback (⇔ Maxwell's demon)

Multiple measurement & feedback (Fujitani & Suzuki, 2010)

$$X_{k+1} = FX_k + GU_k + W_k$$
$$Y_k = HX_k + V_k$$

$$U_k = L_k \hat{X}_k\left(Y_{[1,k]}\right)$$

$$X_k = \tilde{X}_k + \bar{X}_k$$

$$\tilde{X}_{k+1} = F\tilde{X}_k + GU_k$$
$$\tilde{Y}_k = H\tilde{X}_k$$

Deterministic

$$\bar{X}_{k+1} = F\bar{X}_k + W_k$$
$$\bar{Y}_k = H\bar{X}_k + V_k$$

Stochastic (innovation process)

X_k: state
U_k: control input
W_k: thermal noise
Y_k: measurement
V_k: noise

$$\langle \boldsymbol{W} \rangle \geq \Delta \boldsymbol{F} - \boldsymbol{k_B T} \boxed{\boldsymbol{I_c}}$$

$$X_{[1,k]} := (X_1, X_2, \dots, X_k)$$
$$x_{[1,k]} := (x_1, x_2, \dots, x_k)$$

Mutual information between $\bar{X}_{[1,N-1]}$ and $\bar{Y}_{[1,N-1]}$.

Background & Research Objective

Thermodynamics ← Information

Thermodynamic systems with measurements and feedback

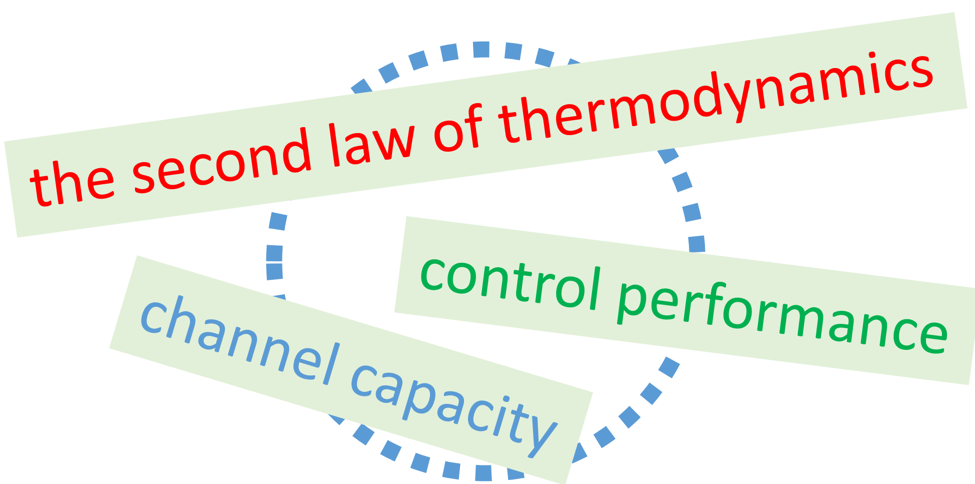
(Sagawa & Ueda, 2010;
Fujitani & Suzuki, 2010; …)

Control theory ← Information

Relationship between channel capacity and control performance

(Wong & Brockett, 1999; Nair & Evans, 2003;
Tsumura & Maciejowski, 2003;
Tatikonda & Mitter, 2004; …)

Thermodynamics ← Information + Control theory

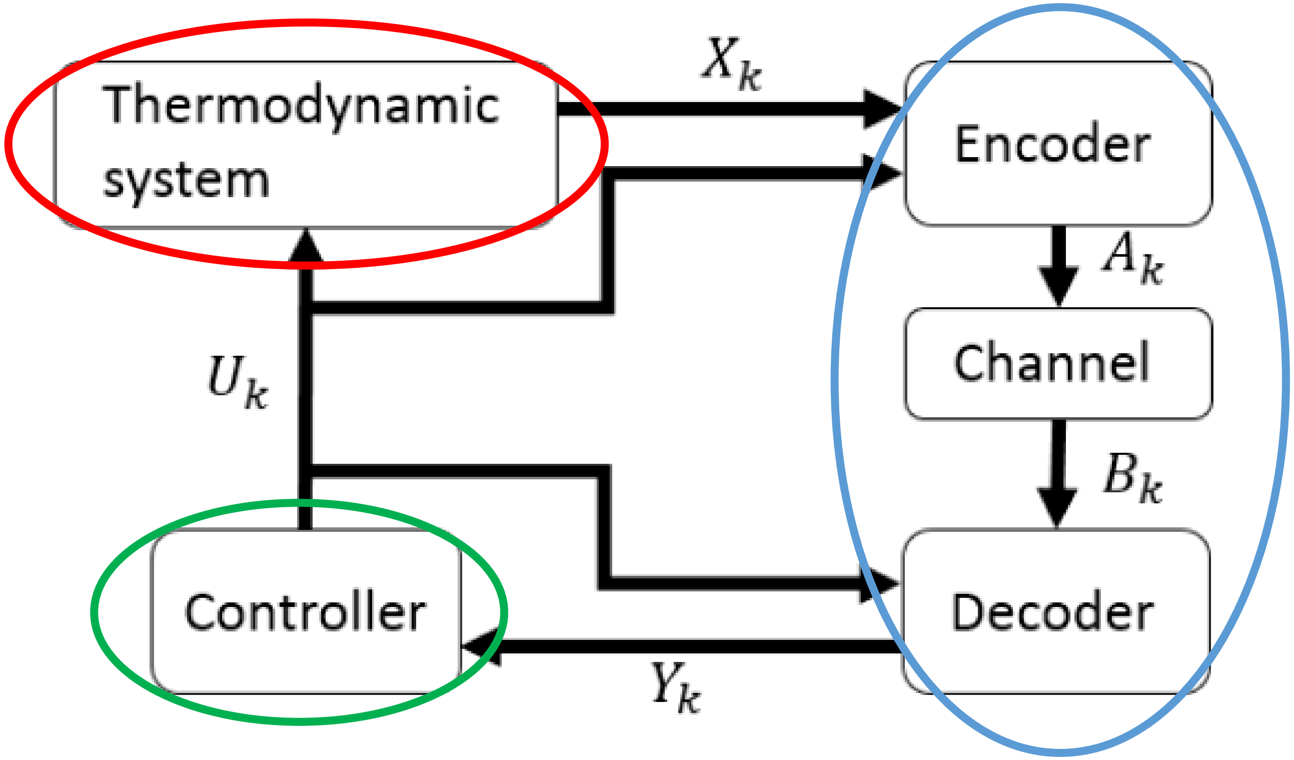


How are they concerned with each other?

Problem Formulation

The Control System

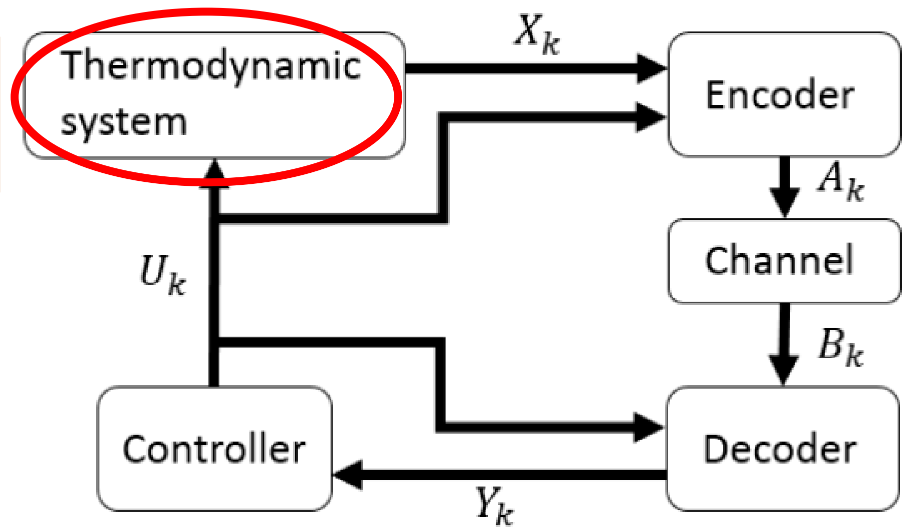
A system governed by thermodynamics Information theory



Control theory

System Dynamics

Upper case letter : random variable
Lower case letter : realization



$$X_{k+1} = FX_k + GU_k + W_k,$$
$$k = 0,1,\cdots,N-1, N \geq 2$$

- N particles with mass m
- State: $x = (r'p')'$
  - $r = (r_1' r_2' \cdots r_N')'$: position vector
  - $p = (p_1' p_2' \cdots p_N')'$: momentum vector

Assumptions:

- $\{W_k\}$(thermal noise): i.i.d sequence of random variables with zero mean.
- X_0 : initial state that satisfies canonical distribution

$$f(x_0) = \frac{e^{-\beta H(x_0;u_0)}}{\int e^{-\beta H(x_0;u_0)}\, dx}$$

$$H(x;u) = \sum_{i=1}^{N} \frac{p_i}{2m} + V(r,u)$$

Hamiltonian

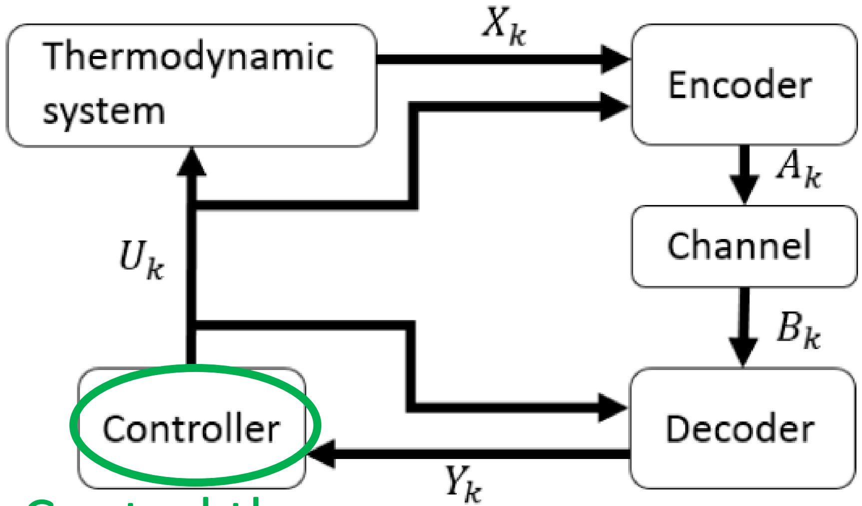
Potential energy

Kinetic energy

$\beta = \frac{1}{k_B T}$: thermodynamic beta

Controller

$$X_{k+1} = FX_k + GU_k + W_k,$$
$$k = 0,1,\cdots,N-1, N \geq 2$$



Control theory

Linear feedback control law:

$$U_k = K_k Y_k$$

Cost function (index of the control performance)

$$J_N = E\left(\sum_{k=1}^{N-1} (X_{k+1} - x_d)' Q (X_{k+1} - x_d) + U_k' S U_k\right)$$

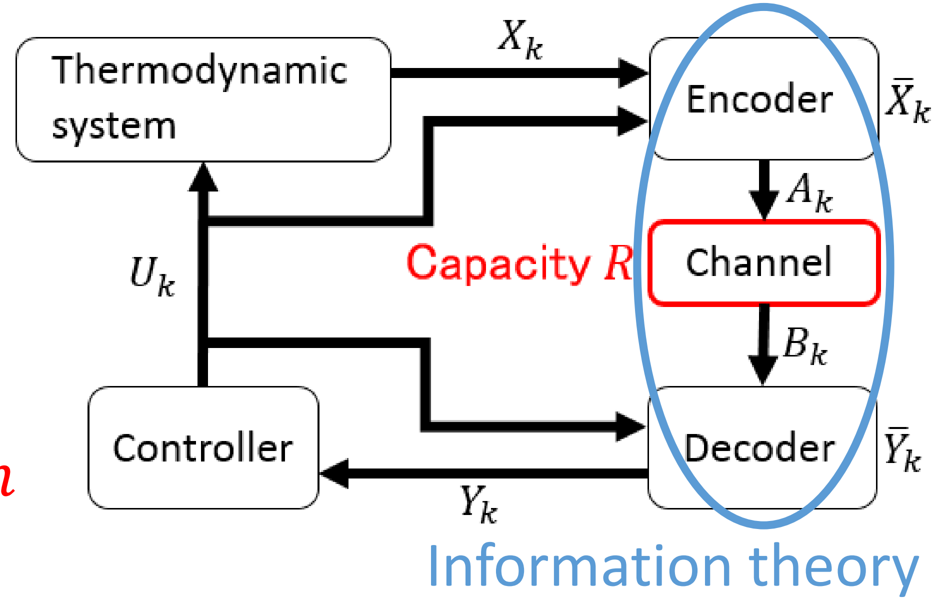
$E(\cdot)$: expected value

x_d: target state

Assumptions: $Q \succcurlyeq 0, S \succ 0, \left(F, Q^{1/2}\right)$ observable

Optimal controller:
The one that minimizes the value of J_N given a fixed communication model.

The Channel Model



- Noiseless digital channel：
  - $a_k = b_k, \forall k$
  - Input alphabet size： m
  - Channel capacity $R = \ln m$

Equi-memory expectation predictive (EMEP) encoder & decoder

(Tatikonda & Mitter, 2004)

- Output of encoder

$$A_k = q\left(\bar{X}_k - E\left(\bar{X}_k | B_{[1,k-1]}\right)\right)$$

  - $q(\cdot)$: quantizer
  - $E(\cdot)$: expected value

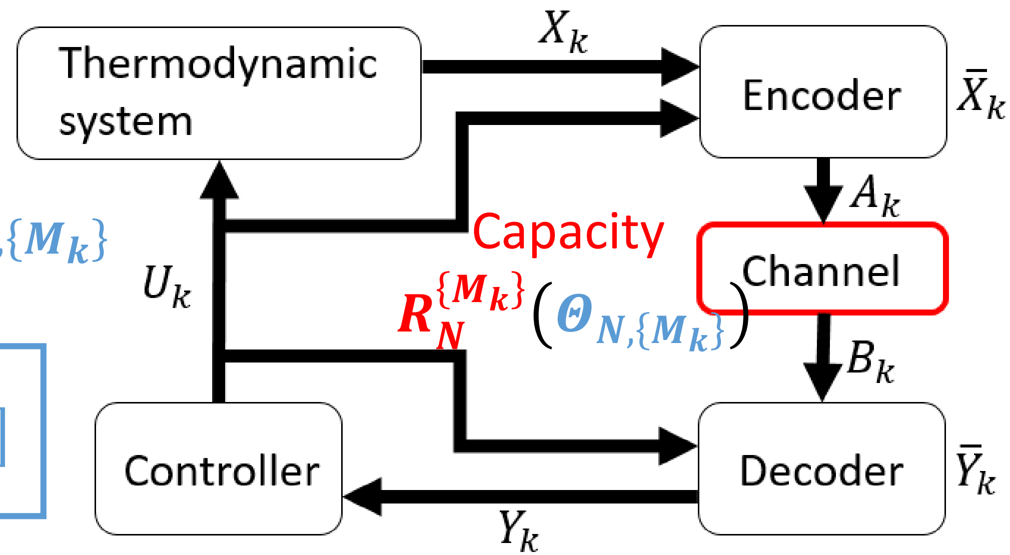
- Output of decoder

$$Y_k = \bar{Y}_k + \sum_{i=0}^{k-i-1} F^{k-i-1} G U_i \,,$$

$$\bar{Y}_k = E\left(\bar{X}_k | B_{[1,k]}\right)$$

$\bar{X}_k, \bar{Y}_k$: the state and the decoder output **that correspond to the innovation process**.

Proper Encoder



estimation error measure $\Theta_{N,\{M_k\}}$

$$J_N = \hat{J}_N + \boxed{\sum_{k=1}^{N-1} E[\Delta_k' M_k \Delta_k]}$$

the optimal control input

the channel capacity & the quantizer in the encoder

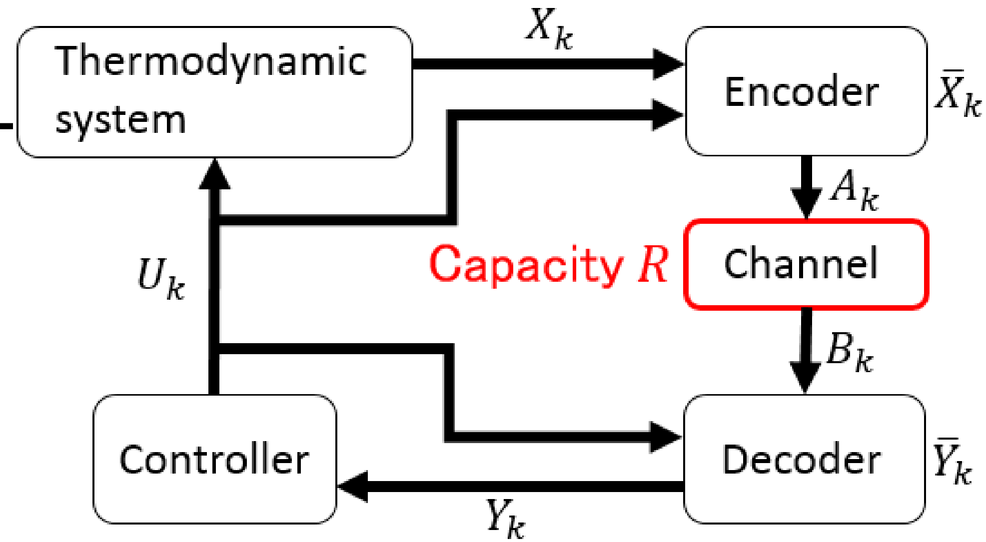
Proper encoder:

the one that realizes a given value of $\Theta_{N,\{M_k\}} (= J_N - \hat{J}_N)$ with ***the least channel capacity*** given the optimal controller.

➡ ***The least channel capacity*** : $R_N^{\{M_k\}}(J_N - \hat{J}_N)$.

Main Results and Discussions

Main Result 1



Theorem

- Temperature T
- Control horizon N
- EMEP encoder & decoder
- Channel capacity R
- Linear feedback control law

$$\langle W \rangle \geq \Delta F - k_B T (N-1) R$$

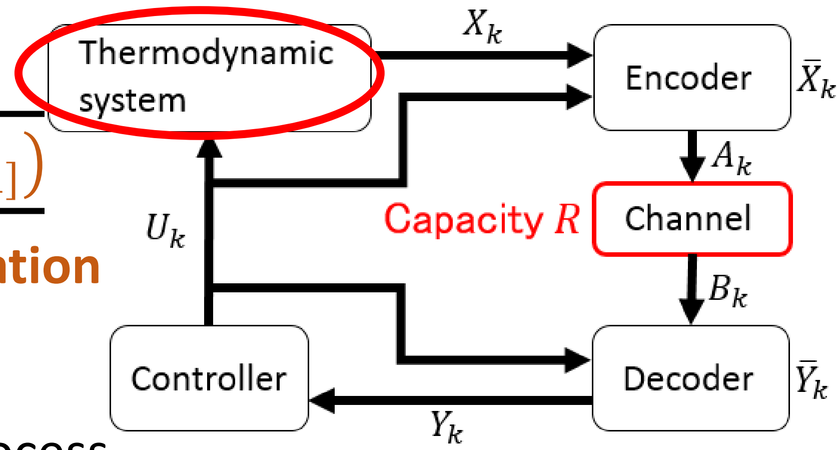
Averaged work done on the system **Free energy difference** **Channel capacity**

Step 1 $\langle W \rangle \geq \Delta F - k_B T I(\bar{X}_{[1,N-1]}; \bar{Y}_{[1,N-1]})$

Step 2 $I(\bar{X}_{[1,N-1]}; \bar{Y}_{[1,N-1]}) = H(\bar{Y}_{[1,N-1]})$

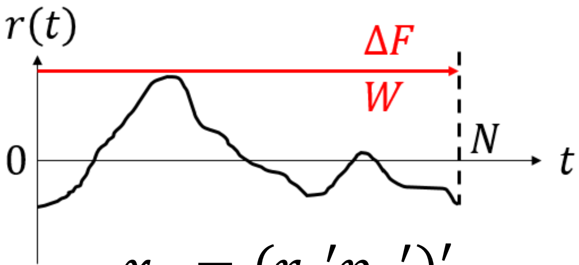
Step 3 $H(\bar{Y}_{[1,N-1]}) \leq (N-1)R$

Proof of Theorem



Step 1 $\langle W\rangle \geq \Delta F - k_B T I\left(\bar{X}_{[1,N-1]};\bar{Y}_{[1,N-1]}\right)$

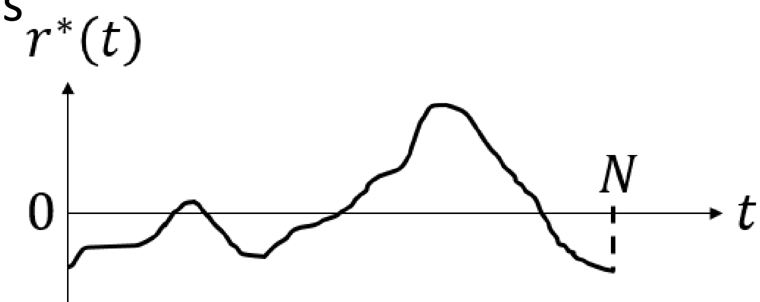
Mutual information



$$x_k = ({r_k}'{p_k}')'$$

$$I\left(\bar{X}_{[1,N-1]}|\bar{Y}_{[1,N-1]}\right) := \ln \frac{f\left(\bar{x}_{[1,N-1]}|\bar{y}_{[1,N-1]}\right)}{f\left(\bar{x}_{[1,N-1]}\right)}$$

Backward process



$${x_k}^* = ({r_{N-k}}' \; -{p_{N-k}}')'$$

${r_k}^*$

Detailed fluctuation theorem (G. E. Crooks, 1998) :

$$e^{\beta(\Delta F - W)} G = \overleftarrow{G}$$

$$G := f(x_0)\prod_{k=0}^{N-1} f(x_{k+1}|x_k;u_k)$$

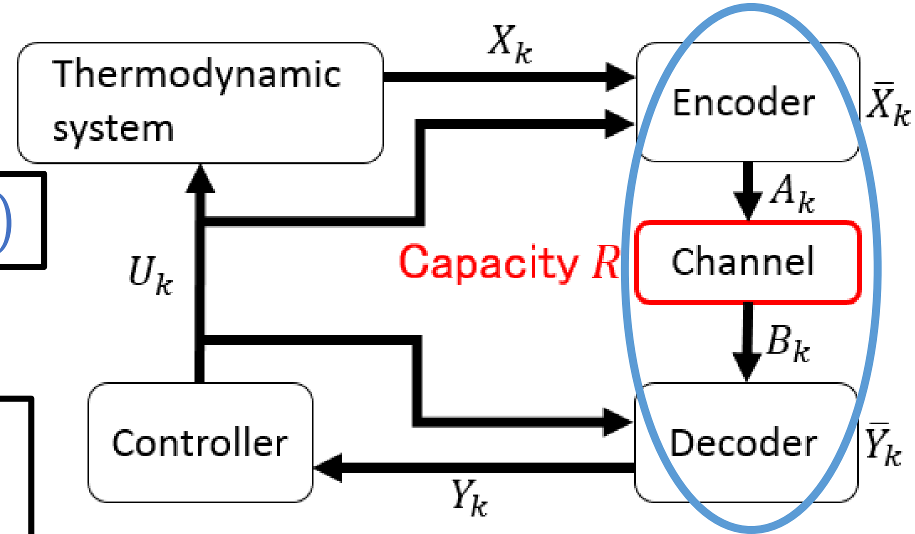
$$\overleftarrow{G} := f(x_N^*)\prod_{k=0}^{N-1} \overleftarrow{f}(x_k^*|x_{k+1}^*;u_k^*)$$

$$1 = e^{\beta\Delta F}\left\langle e^{-\beta W - I\left(\bar{X}_{[1,N-1]}|\bar{Y}_{[1,N-1]}\right)}\right\rangle$$

Jensen's inequality

$$\langle W\rangle \geq \Delta F - k_B T I\left(\bar{X}_{[1,N-1]};\bar{Y}_{[1,N-1]}\right)$$

Proof of Theorem



Step 2 $I\left(\bar{X}_{[1,N-1]};\bar{Y}_{[1,N-1]}\right) = H\left(\bar{Y}_{[1,N-1]}\right)$

Mutual information **Entropy**

EMEP encoder & decoder

- Output of encoder

$$A_k = q\left(\bar{X}_k - E\left(\bar{X}_k | B_{[1,k-1]}\right)\right)$$

  - q: quantizer
  - $E(\cdot)$: expectation

- Output of decoder

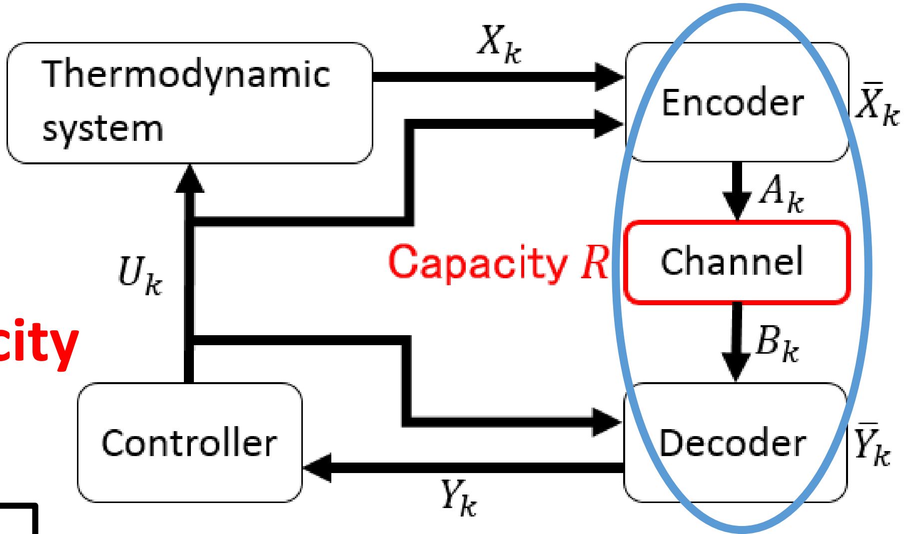
$$\bar{Y}_k = E\left(\bar{X}_k | B_{[1,k]}\right)$$

$\bar{y}_{[1,k]}$ is uniquely determined given $\bar{x}_{[1,k]}$

$$H\left(\bar{Y}_{[1,N-1]} | \bar{X}_{[1,N-1]}\right) = 0$$

$$I\left(\bar{X}_{[1,N-1]};\bar{Y}_{[1,N-1]}\right)$$
$$= H\left(\bar{Y}_{[1,N-1]}\right) - H\left(\bar{Y}_{[1,N-1]} | \bar{X}_{[1,N-1]}\right) = H\left(\bar{Y}_{[1,N-1]}\right)$$

Proof of Theorem



Step 3 $H(\bar{Y}_{[1,N-1]}) \leq (N-1)R$

Entropy

Control horizon

Capacity

EMEP encoder & decoder

◆ Output of decoder

$$\bar{Y}_k = E(\bar{X}_k | B_{[1,k]})$$

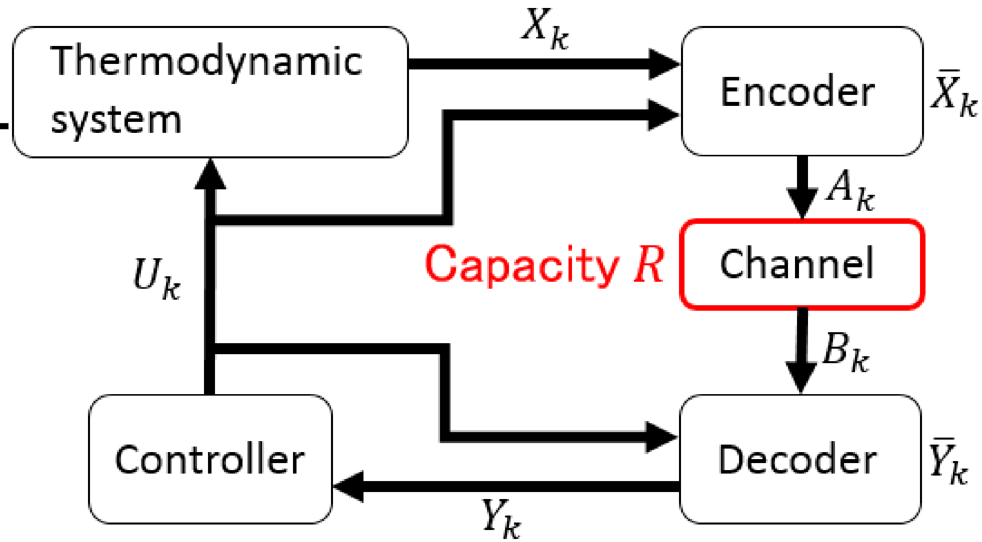
Channel output

$\bar{y}_{[1,k]}$ is uniquely determined given $b_{[1,k]}$

$$H(B_{[1,k]}) \geq H(\bar{Y}_{[1,k]})$$

$$H(\bar{Y}_{[1,k]}) \leq H(B_{[1,k]}) = H(A_{[1,k]}) \leq \sum_{k=1}^{N-1} H(A_k) \leq (N-1)R$$

Discussion



Theorem

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$$\langle W \rangle \geq \Delta F - k_B T (N-1) R$$

Averaged work done on the system

Free energy difference

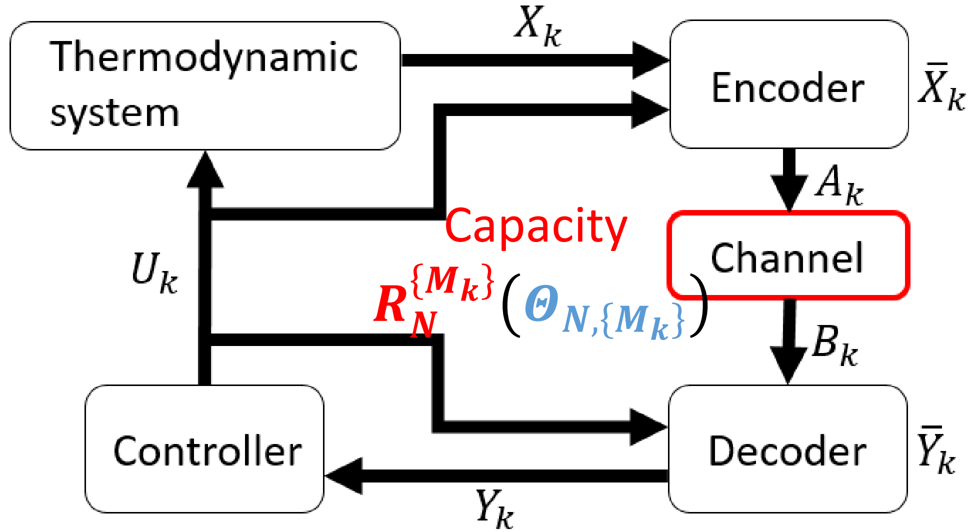
Channel capacity

Given a fixed value of ΔF, as ***MORE*** channel capacity R is used,

- ***LESS work needs to be done*** on the system.
- ***MORE work can be extracted*** from the system.

Main Result 2

$$J_N = \hat{J}_N + \underbrace{\boxed{\sum_{k=1}^{N-1} E[\Delta_k' M_k \Delta_k]}}_{\Theta_{N,\{M_k\}}}$$



Proper encoder:

the one that realizes a given value of $\Theta_{N,\{M_k\}} (= J_N - \hat{J}_N)$ with ***the least channel capacity*** given the optimal controller.

➡ ***The least channel capacity*** : $R_N^{\{M_k\}}(J_N - \hat{J}_N)$.

Corollary

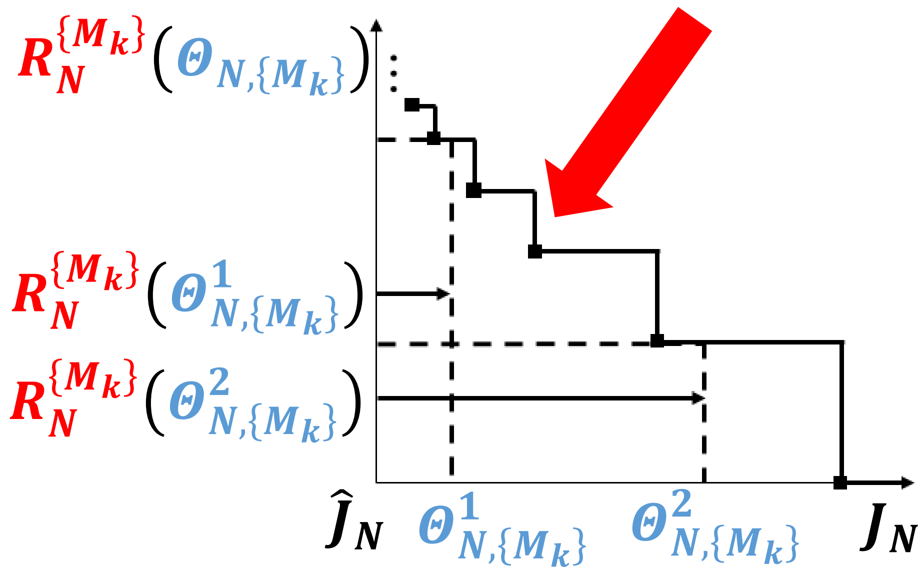
- The ***optimal*** controller
- A ***proper*** EMEP encoder & decoder
- Channel capacity $R_N^{\{M_k\}}(J_N - \hat{J}_N)$

➡ $\langle W \rangle \geq \Delta F - k_B T (N-1) R_N^{\{M_k\}}(J_N - \hat{J}_N)$

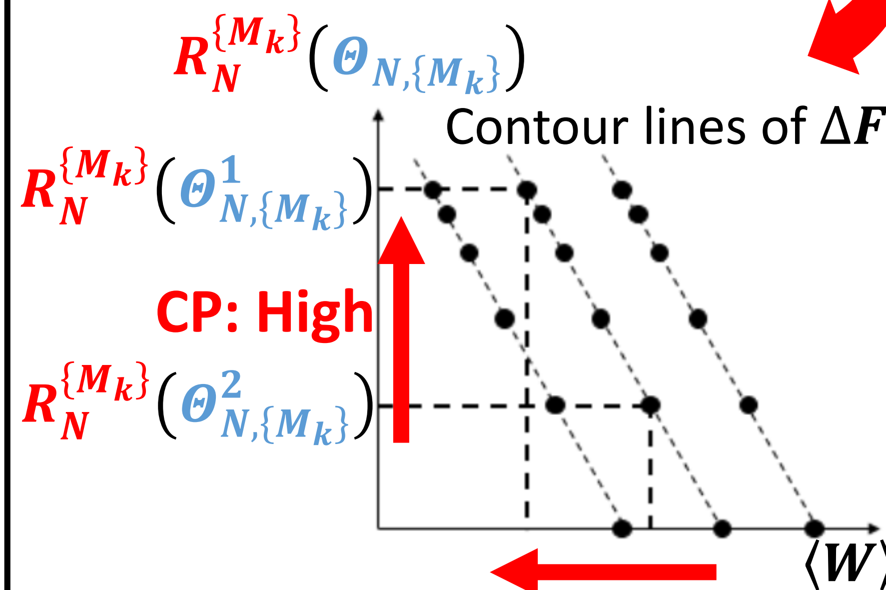
Discussion

$$\langle W\rangle \geq \Delta F - k_B T(N-1)\underline{R}_N(J_N - \hat{J}_N)$$

$R_N^{\{M_k\}}(\Theta_{N,\{M_k\}}) = R_N^{\{M_k\}}(J_N - \hat{J}_N)$ is a decreasing function of $\Theta_{N,\{M_k\}}$.



Control Performance (CP): High



Work: Small

the second law of thermodynamics

channel capacity

control performance

Relationship elucidated!

Conclusions

Conclusions

- Derived the second law containing a term of channel capacity ***under linear feedback control law*** (Main result 1).
- Elucidate the relationship between the control performance, the channel capacity and the second law where ***the optimal controller and a proper encoder are used*** (Main result 2).