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Optimal Multiple Controlling Nodes Problem for Multi-agent Systems via Alt-PageRank

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The Aim of This Research

Systems: Large scaled multi-agent consensus systems (number of agents $n \gg 1$) including a small number (κ) of agents given reference signals.

Problem: Find the optimal choice of a set of agents given the constant reference signals, attaining the fastest convergence rate to the equilibrium of the whole system.

Difficulty: The brute force searching the optimal set requires ${}_nC_\kappa$ times calculations of eigenvalues of $n \times n$ matrices (order ${}_nC_\kappa \times n^3$, e.g., a case of $n = 10^3$, $\kappa = 3$ requires calculations of order 1.7×10^{17}).

Objective: Give a method to find the optimal set of the agents with small computation complexity.

Controlled Consensus Systems

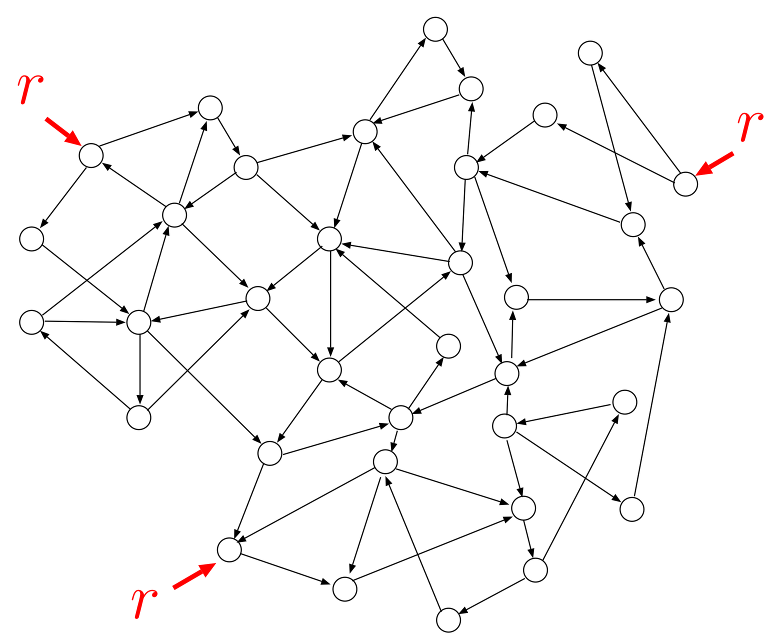

Fig. 1: Large scaled multi-agent consensus system with reference signals

- $\mathcal{G} = (\mathcal{V}, \mathcal{E})$: directed graph
- $\mathcal{V}$: a set of nodes
- $\mathcal{E}$: a set of directed edges
- $(i, j) \in \mathcal{E}$: a directed edge from node $i \in \mathcal{V}$ to node $j \in \mathcal{V}$,
- $\mathcal{N}_i := \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$: the neighbor set which sends information to i
- $\mathcal{B}_i := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$: the neighbor set which receives information from i

Weakly Controlled System (WCS):

normal agents: $\dot{x}_i = \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} (x_j - x_i), \quad \forall i \notin \mathcal{K}$

controlled agents: $\dot{x}_k = \frac{1}{|\mathcal{N}_k|} \sum_{j \in \mathcal{N}_k} (x_j - x_k) + \epsilon(r - x_k), \quad \forall k \in \mathcal{K}$

vector form : $\dot{\boldsymbol{x}} = -\mathcal{L}_\mathcal{K} \boldsymbol{x} + \epsilon r \sum_{k \in \mathcal{K}} \boldsymbol{e}_k, \quad \mathcal{L}_\mathcal{K} := \mathcal{L} + \epsilon \sum_{k \in \mathcal{K}} (\boldsymbol{e}_k \boldsymbol{e}_k^\top)$

- $\mathcal{L}$: the graph Laplacian of the “original” system without the references
- r: reference signal
- $\mathcal{K}$ ($\subseteq \mathcal{V}$): the set of indices of agents to which r is applied
- ϵ: a sufficiently small positive number
- $\boldsymbol{e}_k$: k-th unit vector $\boldsymbol{e}_k = [0 \cdots 0\,1\,0 \cdots 0]^\top$

convergence rate $\Leftarrow$ the slowest eigenvalue of $-\mathcal{L}_\mathcal{K}$
$\Leftarrow$ choice of the controlled agent set $\mathcal{K}$

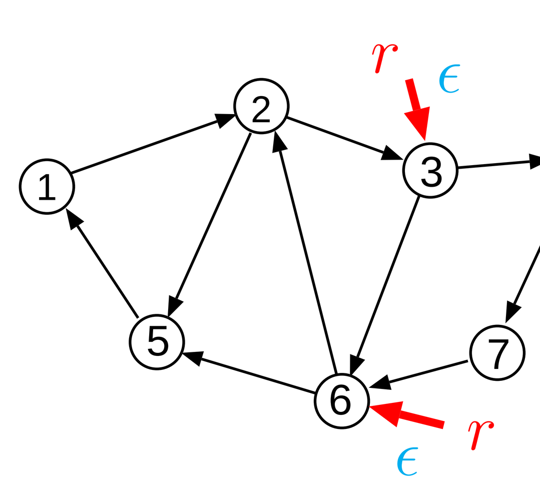

Fig. 2: Example of WCS

$$\mathcal{L}_{\{3,6\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1/2 & 1 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & -1 & 1+\epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 & 1 & -1/2 & 0 \\ 0 & 0 & -1/2 & 0 & 0 & 1+\epsilon & -1/2 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

Problem: Find the set of controlled agents $\mathcal{K}$ with which the slowest eigenvalue of $-\mathcal{L}_\mathcal{K}$ to be fastest.

PageRank and Alt-PageRank

PageRank (Brin & Page 1998):

$$q_i = \sum_{j \in \mathcal{N}_i} \frac{q_j}{|\mathcal{B}_j|}, \quad \boldsymbol{q} = [q_1\ q_2 \cdots q_n]^\top, \quad \boldsymbol{q}^\top = \boldsymbol{q}^\top \Xi, \quad \Xi_{ij} = \begin{cases} \frac{1}{|\mathcal{B}_i|} & \text{if } j \in \mathcal{B}_i \\ 0 & \text{otherwise} \end{cases}$$

Alt-PageRank (Yamamoto & Tsumura 2011):

$$p_i = \sum_{j \in \mathcal{B}_i} \frac{p_j}{|\mathcal{N}_j|}, \quad \boldsymbol{p} = [p_1\ p_2 \cdots p_n]^\top, \quad \boldsymbol{p}^\top = \boldsymbol{p}^\top \Pi, \quad \Pi_{ij} = \begin{cases} \frac{1}{|\mathcal{N}_i|} & \text{if } j \in \mathcal{N}_i \\ 0 & \text{otherwise} \end{cases}$$

$$\boldsymbol{p}^\top \mathcal{L} = \boldsymbol{0}^\top, \quad \mathcal{L} = I - \Pi$$

Remark: Calculation of Alt-PageRank p is order n^3.

Main Results

Theorem: For a WCS, assume that $\epsilon > 0$ is sufficiently small and an integer $\kappa > 0$ is given. Then the convergence rate of WCS with a choice of node set $\mathcal{K}$ satisfying $|\mathcal{K}| = \kappa$ becomes fastest when we choose κ nodes of the largest order in the corresponding Alt-PageRank $\boldsymbol{p}$.

Remark: The result is given by showing the following:

$$\underline{\lambda}(\mathcal{L}_\mathcal{K}) = \epsilon \sum_{k \in \mathcal{K}} p_k + \mathcal{O}(\epsilon^2)$$

Remark: The computation complexity to solve the optimization via Alt-PageRank is $1/{}_nC_\kappa$ of that by the brute force searching (e.g., a case of $n = 10^3$, $\kappa = 3$ via Alt-PageRank requires calculations of order $n = 10^9$. Compare with the order 1.7×10^{17} by the brute force searching).

Numerical Simulations

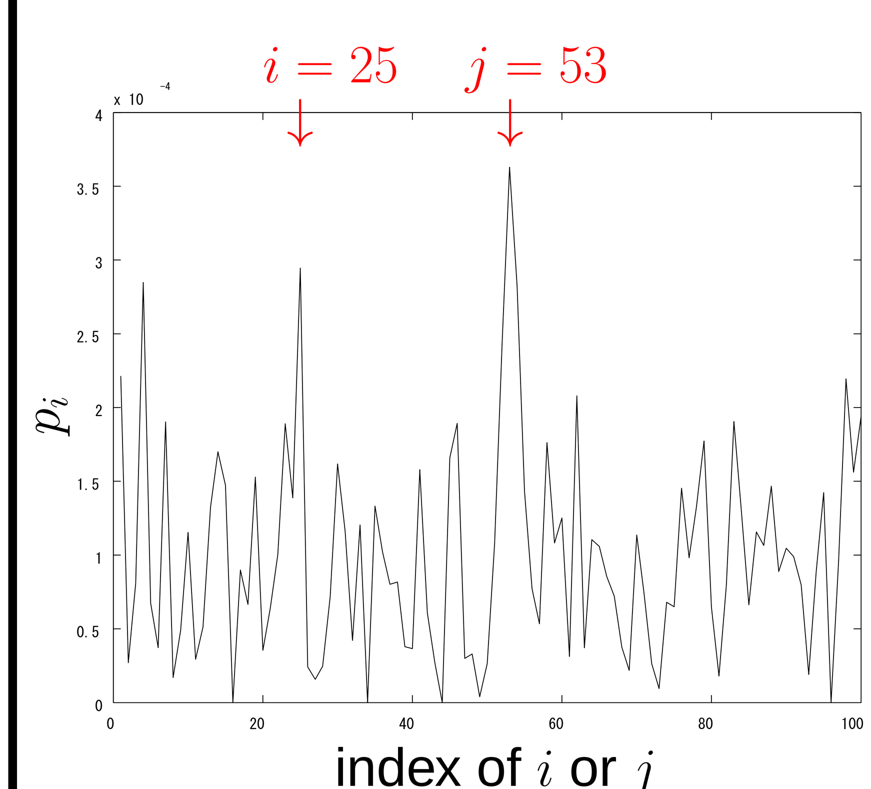

Fig. 3: Values of p_i

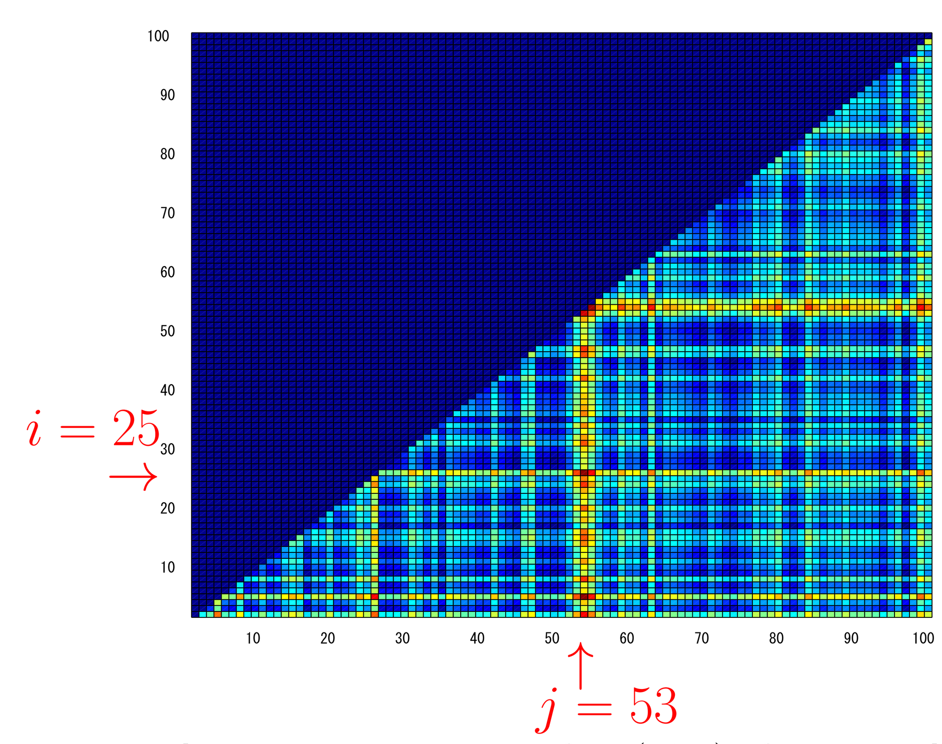

Fig. 4: Values of $\underline{\lambda}(\mathcal{L}_\mathcal{K})$ for each choice of (i, j)

Remark: Fig. 3 ($n = 100$, $\kappa = 2 = |\mathcal{K}|$) shows that the largest and the second p_i are p_{53} and p_{25}, respectively. On the other hand, Fig. 4 shows the values of $\underline{\lambda}(\mathcal{L}_\mathcal{K})$ for each choice of (i, j) for $\mathcal{K}$ (a dark blue rectangular represents number 0 and a dark red rectangular represents a larger number). It also shows that the choice of $(i, j) = (25, 53)$ gives the largest $\underline{\lambda}(\mathcal{L}_\mathcal{K})$ and it coincides with the above mentioned order of p_i.

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http://www.cyb.ipc.i.u-tokyo.ac.jp/members/tsumu/index-e.html