Optimal Control/Observation Points Problem and Separation Principle of Weakly Controlled Large-scaled Multi-agent Systems

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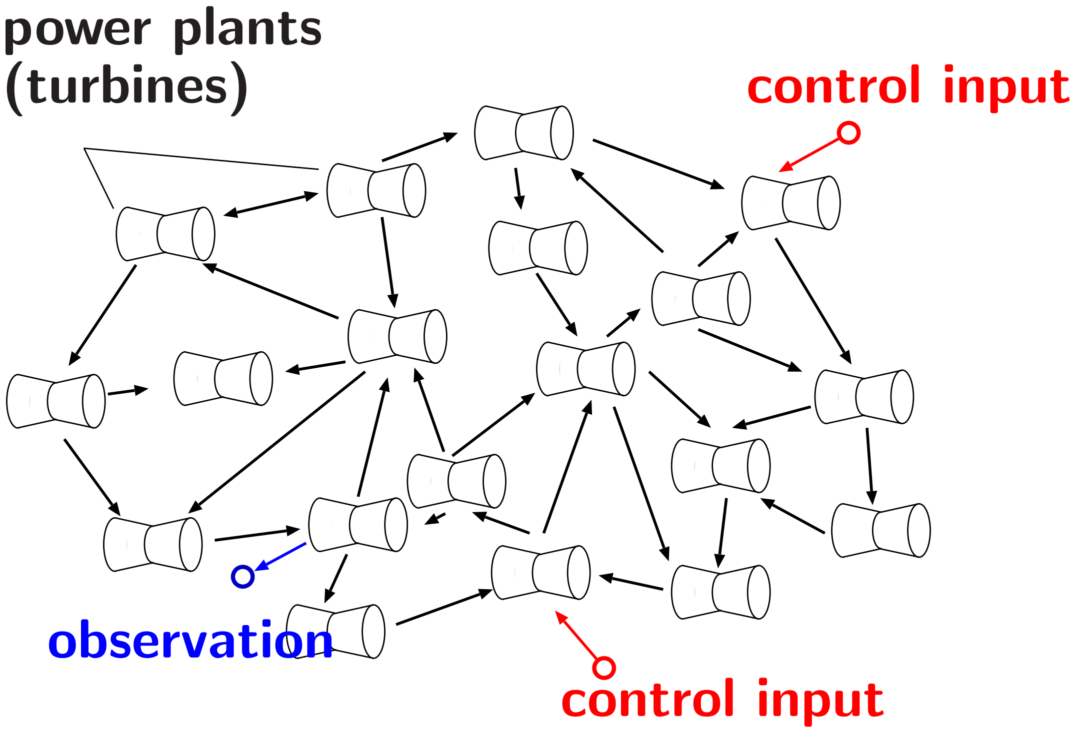
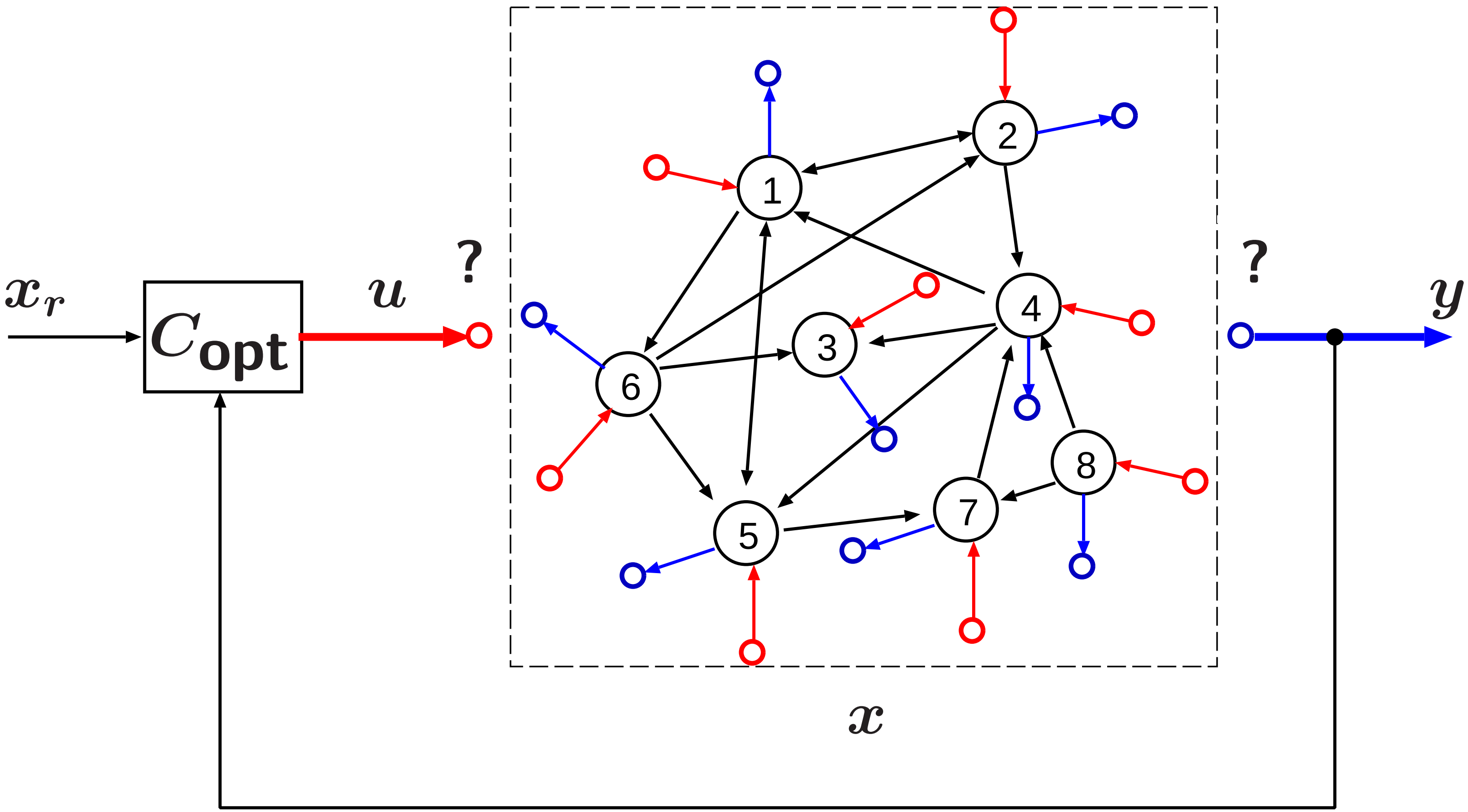
Motivation

large-scaled electric power network

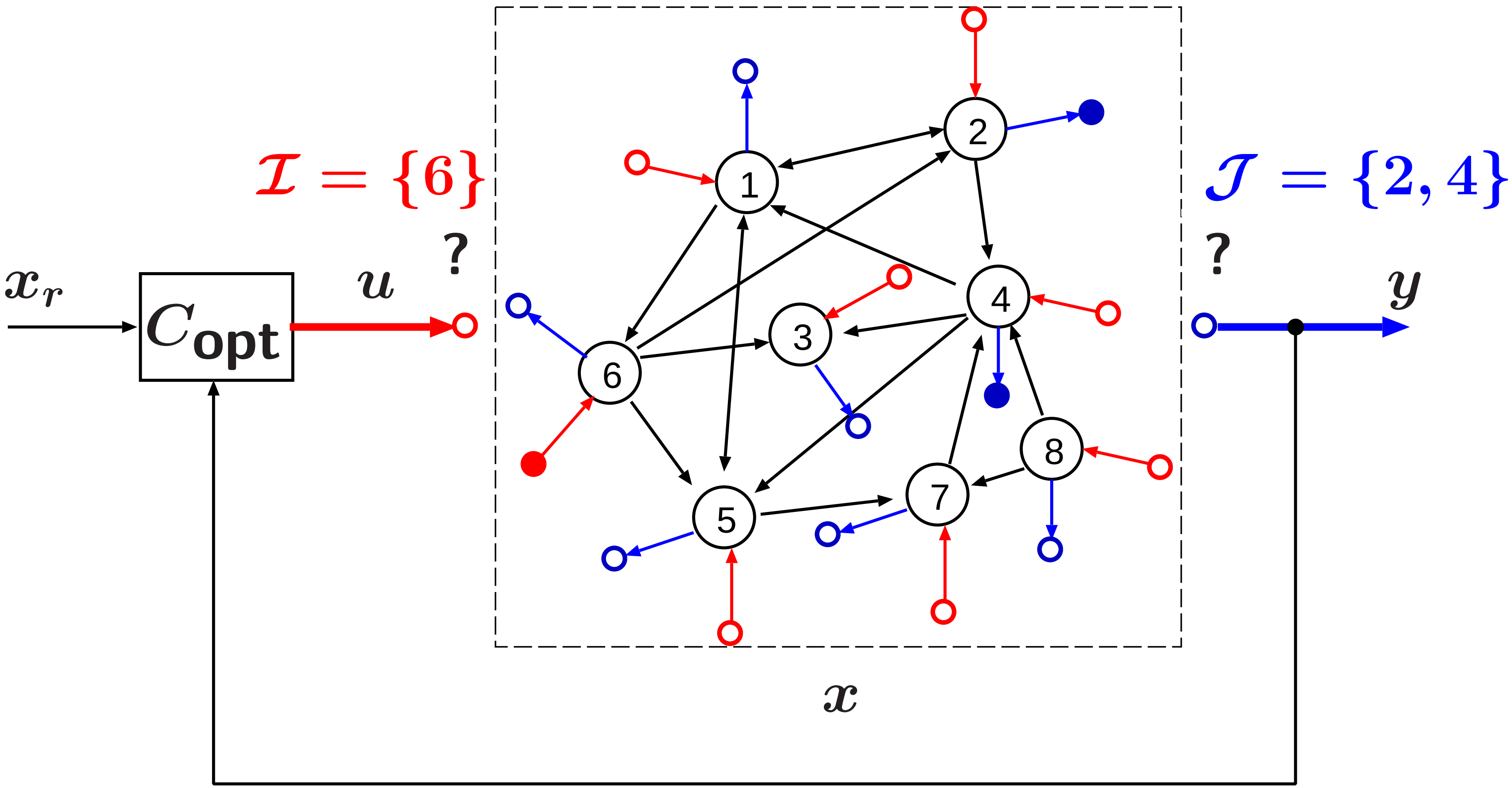


- synchronize/regulate turbine frequencies
- the number of directly controlled/observed turbines is restricted
- magnitude of control input to each turbine is restricted

"Which power plants should be controlled/observed ?"



output feedback controlled synchronization



output feedback controlled synchronization

combinatorial number for selecting $\mathcal{I}$ and $\mathcal{J}$ is huge

Local Dynamics

multi-agent system composed of N dynamical subsystems:

$$\begin{aligned} \dot{x}_k(t) &= A_h x_k(t) + b_n u_k(t) \\ y_k(t) &= c_h x_k(t),\ k = 1, 2, \ldots, N \end{aligned}$$

$u_k(t)$, $y_k(t)$ ($\in \mathbb{R}$), $x_k(t)$ ($\in \mathbb{R}^\nu$)

Network (global) Structure

A: network matrix to represent the connection between the subsystems

$\mathcal{I}$ ($|\mathcal{I}| = M_{\rm i}$): the set of agents with control input, "control points"

$\mathcal{J}$ ($|\mathcal{J}| = M_{\rm o}$): the set of observed agents, "observation points"

$$\mathcal{I} = \{i_1,\ i_2,\ \ldots\ ,\ i_{M_{\rm i}}\},\ \mathcal{J} = \{j_1,\ j_2,\ \ldots\ ,\ j_{M_{\rm o}}\}$$

$B_{\mathcal{I}}$, $C_{\mathcal{J}}$: represent the set of control/observation points:

$$B_{\mathcal{I}} = \begin{bmatrix} e_{i_1} & e_{i_2} & \cdots & e_{i_{M_{\rm i}}} \end{bmatrix} \in \mathbb{R}^{N \times M_{\rm i}}$$

$$C_{\mathcal{J}} = \begin{bmatrix} e_{j_1} & e_{j_2} & \cdots & e_{j_{M_{\rm o}}} \end{bmatrix}^\top \in \mathbb{R}^{M_{\rm o} \times N}$$

$$e_i = [0 \ \cdots \ 0 \ \underbrace{1}_{i\text{–th}} \ 0 \ \cdots \ 0]^\top$$

The Whole System

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}_{\mathcal{I}}u(t) + v_1(t)$$
$$y(t) = \mathcal{C}_{\mathcal{J}}x(t) + v_2(t)$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} \in \mathbb{R}^{\nu N}, \;\; u(t) \in \mathbb{R}^{M_\mathrm{i}}, \;\; y(t) \in \mathbb{R}^{M_\mathrm{o}}$$

$$\mathcal{A} = I \otimes A_h + A \otimes (b_h c_h)$$
$$\mathcal{B}_{\mathcal{I}} = B_{\mathcal{I}} \otimes b_h, \;\; \mathcal{C}_{\mathcal{J}} = C_{\mathcal{J}} \otimes c_h$$

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}_{\mathcal{I}}u(t) + v_1(t)$$
$$y(t) = \mathcal{C}_{\mathcal{J}}x(t) + v_2(t)$$

$v_1(t)$: **disturbance,** $v_2(t)$: **observation noise**

$\left[v_1^\top(t)\ v_2^\top(t)\right]^\top$: **white Gaussian with covariance matrix**

$$\begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} = \begin{bmatrix} w_1 I & 0 \\ 0 & w_2 I \end{bmatrix},\ w_1 > 0,\ w_2 > 0$$

Assumption

A, A_h have stable distinct eigenvalues except for a single zero eigenvalue.

x_r: reference signal for $x(t)$

$$x_r = \mathbf{p_r} \otimes \mathbf{q_r}$$

$\mathrm{p_l}^\top$, $\mathbf{p_r}$: the left/right zero eigenvectors of A

$\mathrm{q_l}^\top$, $\mathbf{q_r}$: the left/right zero eigenvectors of A_h

Problem Formulation

control purpose

(I) $x_e(t) := x(t) - x_r$, x_r: **reference**

$$\mathbf{E}\left[x_e(\infty)\right] = \mathbf{E}\left[x(\infty) - x_r\right] = 0$$

(II) minimize J **given by**

$$J := \lim_{t_\mathrm{f}\to\infty} \frac{1}{t_\mathrm{f}} \mathbf{E}\left[\int_0^{t_\mathrm{f}} x_e^\top(t) Q x_e(t) + u^\top(t) R u(t) dt\right]$$

$$Q := qI,\ q > 0, \quad R := rI,\ r > 0$$

Assumption

> $r \gg q$ **(controlled by a weak input compared to the size of the whole system)**
> $w_2 \gg w_1$ **(estimation by a weak feedback compared to the size of the whole system).**

Optimal Control/Observation Points Problem in Output Feedback Systems

Problem

[Optimal Control/Observation Points Problem in Output Feedback Systems] For given $M_{\rm i}$ (# control points), $M_{\rm o}$ (# observation points), x_r, find $\mathcal{I}$ and $\mathcal{J}$ s.t. J in (II) is minimized under (I) with the optimal controller from y to u.

- The optimal J^* depends on $\mathcal{I}$ and $\mathcal{J}$:

$$J^*(\mathcal{I},\mathcal{J}) = w_1 \mathrm{tr}(\boldsymbol{X}_{\mathcal{I}}) + 2\mathrm{tr}(\mathcal{A}\boldsymbol{Y}_{\mathcal{J}}\boldsymbol{X}_{\mathcal{I}}) + q\mathrm{tr}(\boldsymbol{Y}_{\mathcal{J}})$$
$$\mathcal{A}^\top \boldsymbol{X}_{\mathcal{I}} + \boldsymbol{X}_{\mathcal{I}}\mathcal{A} - \boldsymbol{X}_{\mathcal{I}}\mathcal{B}_{\mathcal{I}}R^{-1}\mathcal{B}_{\mathcal{I}}{}^\top \boldsymbol{X}_{\mathcal{I}} + D^\top Q D = O$$
$$\mathcal{A}\boldsymbol{Y}_{\mathcal{J}} + \boldsymbol{Y}_{\mathcal{J}}\mathcal{A}^\top - \boldsymbol{Y}_{\mathcal{J}}\mathcal{C}_{\mathcal{J}}{}^\top W_2^{-1}\mathcal{C}_{\mathcal{J}}\boldsymbol{Y}_{\mathcal{J}} + W_1 = O$$

- Our problem is to find the optimal pair of $\mathcal{I}$ and $\mathcal{J}$ which minimizes J^*.

- It is solvable by a brute-force searching of all the possible pairs of $\mathcal{I}$ and $\mathcal{J}$ in principle.
- However, such brute-force searching is actually infeasible from computation complexity when the number of agents is large (combinatorial number for selecting $\mathcal{I}$ and $\mathcal{J}$ is huge).
- The computation complexity is in order $\binom{N}{M_{\rm i}} \cdot \binom{N}{M_{\rm o}} \cdot O(N^3\nu^3)$.

example: $N = 10^4$, $\nu = 3$, $M_{\rm i} = 5$, and $M_{\rm o} = 5$

$\Rightarrow$

$$\binom{N}{M_{\rm i}} \cdot \binom{N}{M_{\rm o}} \cdot \mathcal{O}(N^3\nu^3) = (8.3 \times 10^{17})^2 \times \mathcal{O}(N^3\nu^3)$$
$$\sim 1.86 \times 10^{52}$$

"Can we solve this difficulty?"

State Feedback Case

Problem

[Optimal Control Points Problem in State Feedback Systems]
For given M_{i} (# control points) and x_r, find $\mathcal{I}$ (control points) with which $J^*(\mathcal{I})$ in (II) is minimized under (I) with the optimal state feedback from x to u.

Proposition

$$J^*(\mathcal{I}) = \mathrm{tr}\left(\boldsymbol{X}_{\mathcal{I}} \boldsymbol{W}_1\right)$$

State Estimation Problem

$$\text{cost function:} \quad J_e := \mathsf{E}\left[(x(t) - \hat{x}(t))^\top (x(t) - \hat{x}(t))\right]$$

$\hat{x}(t)$: the state estimation of $x(t)$

Problem

[Optimal Observation Points Problem in State Estimation] For given $\mathcal{I}$ and M_{o} (# observation points), find $\mathcal{J}$ (observation points) with which $J_e^*(\mathcal{J})$ is minimized by the corresponding optimal state estimator of $x(t)$.

Proposition

$$J_e^*(\mathcal{J}) = \mathrm{tr}\,(Y_{\mathcal{J}})$$

Optimal Control/Observation Points Problem in Output Feedback Systems

$$p_{qr} := \frac{r}{q} \gg 1, \ p_{w_1 w_2} := \frac{w_2}{w_1} \gg 1$$

$$\Uparrow$$

Assumption

$r \gg q$ (controlled by a weak input compared to the size of the whole system)
$w_2 \gg w_1$ (estimation by a weak feedback compared to the size of the whole system).

Theorem

There exist $\bar{p}_{qr} > 0$ and $\bar{p}_{w_1 w_2} > 0$ and when $p_{qr} > \bar{p}_{qr}$, $p_{w_1 w_2} > \bar{p}_{w_1 w_2}$,

the optimal pair of the sets of control/observation points $\mathcal{I}^*$ and $\mathcal{J}^*$ which minimizes $J^*(\mathcal{I}, \mathcal{J})$ is given as follows:

$$\mathcal{I}^* = \Big\{ i_1, i_2, \ldots, i_{M_\mathrm{i}} \Big| \, |[\mathrm{p_l}]_{i_1}| \geq |[\mathrm{p_l}]_{i_2}| \geq \cdots \geq \Big|[\mathrm{p_l}]_{i_{M_\mathrm{i}}}\Big| \geq \Big|[\mathrm{p_l}]_{i_{M_\mathrm{i}+1}}\Big| \geq \Big|[\mathrm{p_l}]_{i_{M_\mathrm{i}+2}}\Big| \geq \cdots \geq |[\mathrm{p_l}]_{i_N}| \Big\}$$

$$\mathcal{J}^* = \Big\{ j_1, j_2, \ldots, j_{M_\mathrm{o}} \Big| \, |[\mathrm{p_r}]_{j_1}| \geq |[\mathrm{p_r}]_{j_2}| \geq \cdots \geq \Big|[\mathrm{p_r}]_{j_{M_\mathrm{o}}}\Big| \geq \Big|[\mathrm{p_r}]_{j_{M_\mathrm{o}+1}}\Big| \geq \Big|[\mathrm{p_r}]_{j_{M_\mathrm{o}+2}}\Big| \geq \cdots \geq |[\mathrm{p_r}]_{j_N}| \Big\}$$

where

$\mathrm{p_l}^\top$: left zero eigenvector of A

$\mathrm{p_r}$: right zero eigenvector of A

by employing perturbation theory for eigenvlues/eigenvectors

(Kato (1966), Knopp (1996), Abrachenkov & Haviv (2003), Gohberg & Rodman (1981))

Theorem

$$
\begin{aligned}
&J^*(\mathcal{I},\, \mathcal{J}) \\
&= w_1 q \left((\alpha^X_{\tilde{N}})^{\frac{1}{2}} (\beta^X_{\mathcal{I}\tilde{N}})^{-\frac{1}{2}} \alpha^Y_{\tilde{N}} {p_{qr}}^{\frac{1}{2}} + (\alpha^Y_{\tilde{N}})^{\frac{1}{2}} (\beta^Y_{\mathcal{J}\tilde{N}})^{-\frac{1}{2}} \alpha^X_{\tilde{N}} {p_{w_1 w_2}}^{\frac{1}{2}} \right. \\
&\quad \left. + \mathcal{O}\left({p_{qr}}^0\right) + \mathcal{O}\left({p_{w_1 w_2}}^0\right) + \mathcal{O}\left({p_{qr}}^0 {p_{w_1 w_2}}^0\right) \right)
\end{aligned}
$$

$$
\mathcal{A} = T \Lambda(\mathcal{A}) T^{-1}, \quad
\begin{aligned}
\beta^X_{\mathcal{I}\tilde{N}} &:= \left[T^{-1}\right]_{\tilde{N},:} \mathcal{B}_{\mathcal{I}} {\mathcal{B}_{\mathcal{I}}}^\top \left[T^{-\top}\right]_{:,\tilde{N}} \\
\beta^Y_{\mathcal{J}\tilde{N}} &:= \left[T^\top\right]_{\tilde{N},:} {\mathcal{C}_{\mathcal{J}}}^\top \mathcal{C}_{\mathcal{J}} \left[T\right]_{:,\tilde{N}}
\end{aligned}
$$

Theorem

$$
\begin{aligned}
&J^*(\mathcal{I}, \mathcal{J}) \\
&= w_1 q \left((\alpha^{X}_{\tilde{N}})^{\frac{1}{2}} (\beta^{X}_{\mathcal{I}\tilde{N}})^{-\frac{1}{2}} \alpha^{Y}_{\tilde{N}} {p_{qr}}^{\frac{1}{2}} + (\alpha^{Y}_{\tilde{N}})^{\frac{1}{2}} (\beta^{Y}_{\mathcal{J}\tilde{N}})^{-\frac{1}{2}} \alpha^{X}_{\tilde{N}} {p_{w_1 w_2}}^{\frac{1}{2}} \right. \\
&\quad \left. + \mathcal{O}\left({p_{qr}}^{0}\right) + \mathcal{O}\left({p_{w_1 w_2}}^{0}\right) + \mathcal{O}\left({p_{qr}}^{0} {p_{w_1 w_2}}^{0}\right) \right)
\end{aligned}
$$

example: $N = 10^4$, $\nu = 3$, $M_{\mathrm{i}} = 5$, and $M_{\mathrm{o}} = 5$

brute force search:

$$
\binom{N}{M_{\mathrm{i}}} \cdot \binom{N}{M_{\mathrm{o}}} \cdot \mathcal{O}(N^3 \nu^3) \sim 1.86 \times 10^{52}
$$

$$
\Rightarrow O(N^3) = 1.0 \times 10^{12}
$$

Separation Principle

Optimal Control Points Problem in State Feedback Systems

Corollary

There exists $\bar{p}_{qr} > 0$ and when $\boldsymbol{p_{qr}} > \bar{p}_{qr}$, the optimal set of control points $\mathcal{I}^{i*}$ which minimize $J^*(\mathcal{I})$ is given by $\mathcal{I}^{i*} = \mathcal{I}^*$.

Optimal Observation Points Problem in State Estimation

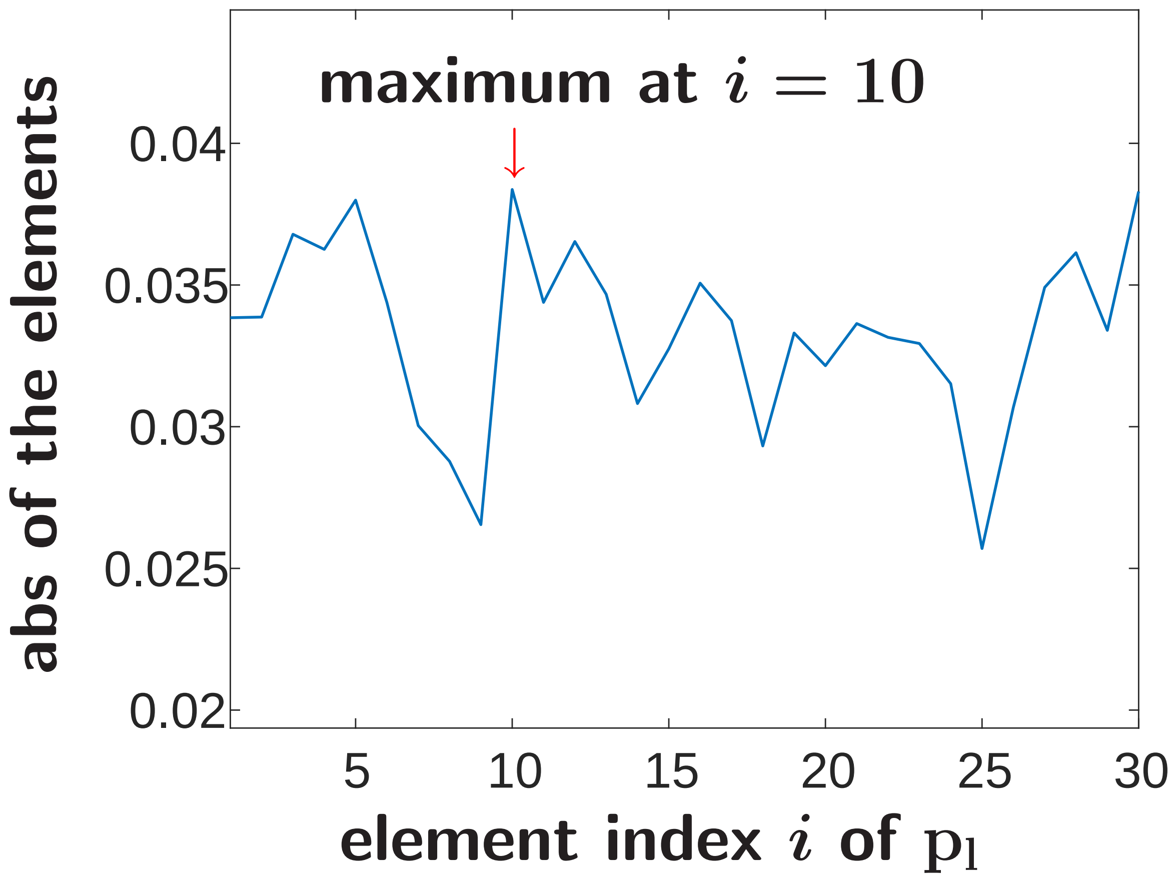
Corollary

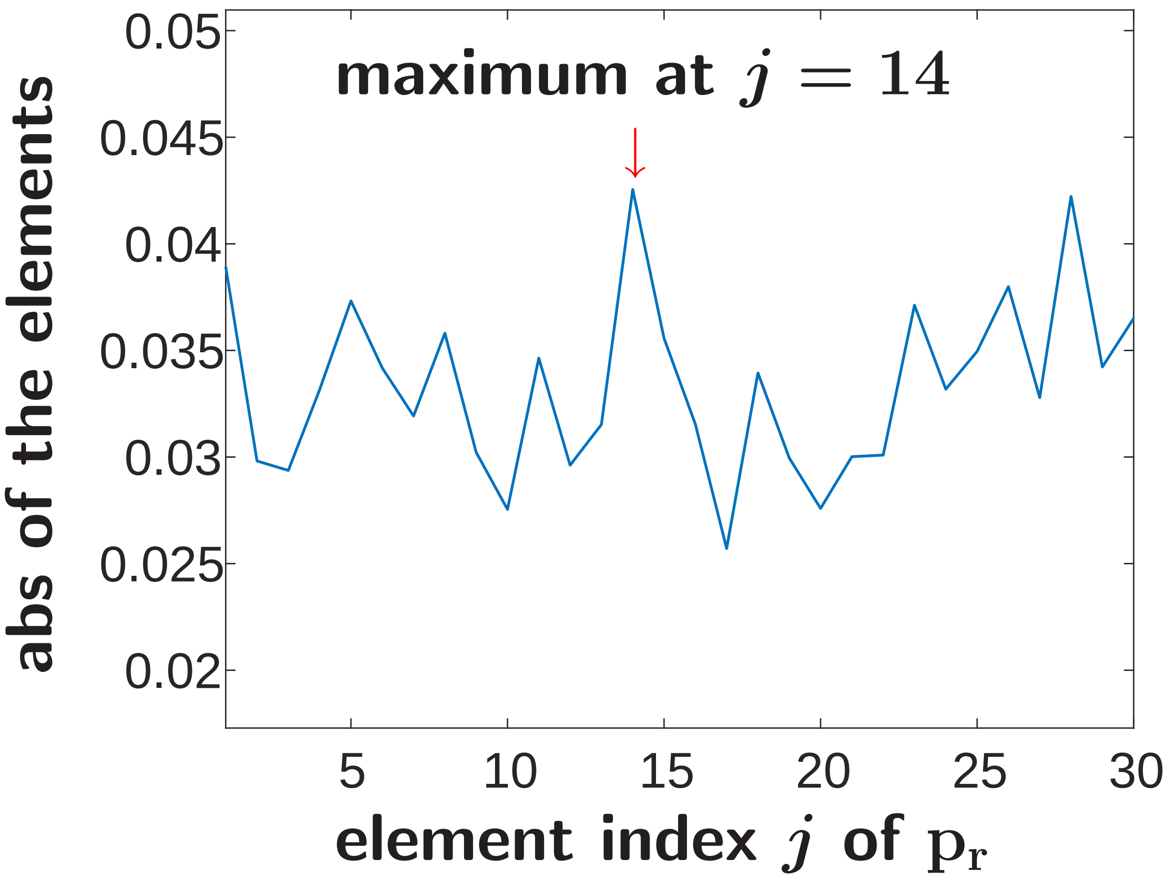
There exists $\bar{p}_{w_1w_2} > 0$ and when $\boldsymbol{p_{w_1w_2}} > \bar{p}_{w_1w_2}$, the optimal set of observation points $\mathcal{J}^{o*}$ which minimize $J_e^*(\mathcal{J})$ is given by $\mathcal{J}^{o*} = \mathcal{J}^*$.

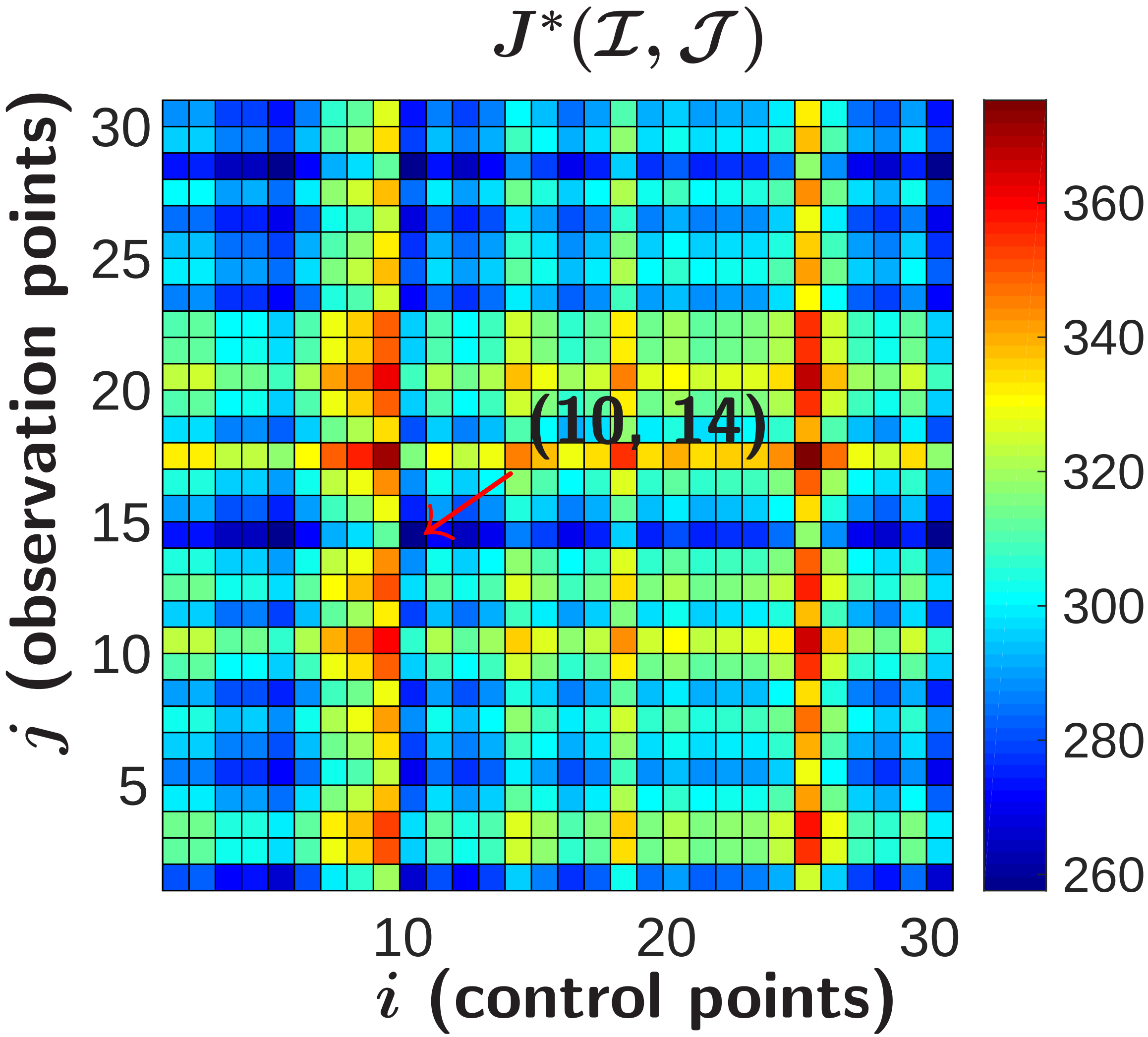
Numerical Simulation

$N = 30$, $\nu = 3$, $q_o = 1$, $r = 100$, $w_1 = 1$, $w_2 = 100$

$M_\mathrm{i} = 1$, $M_\mathrm{o} = 1$

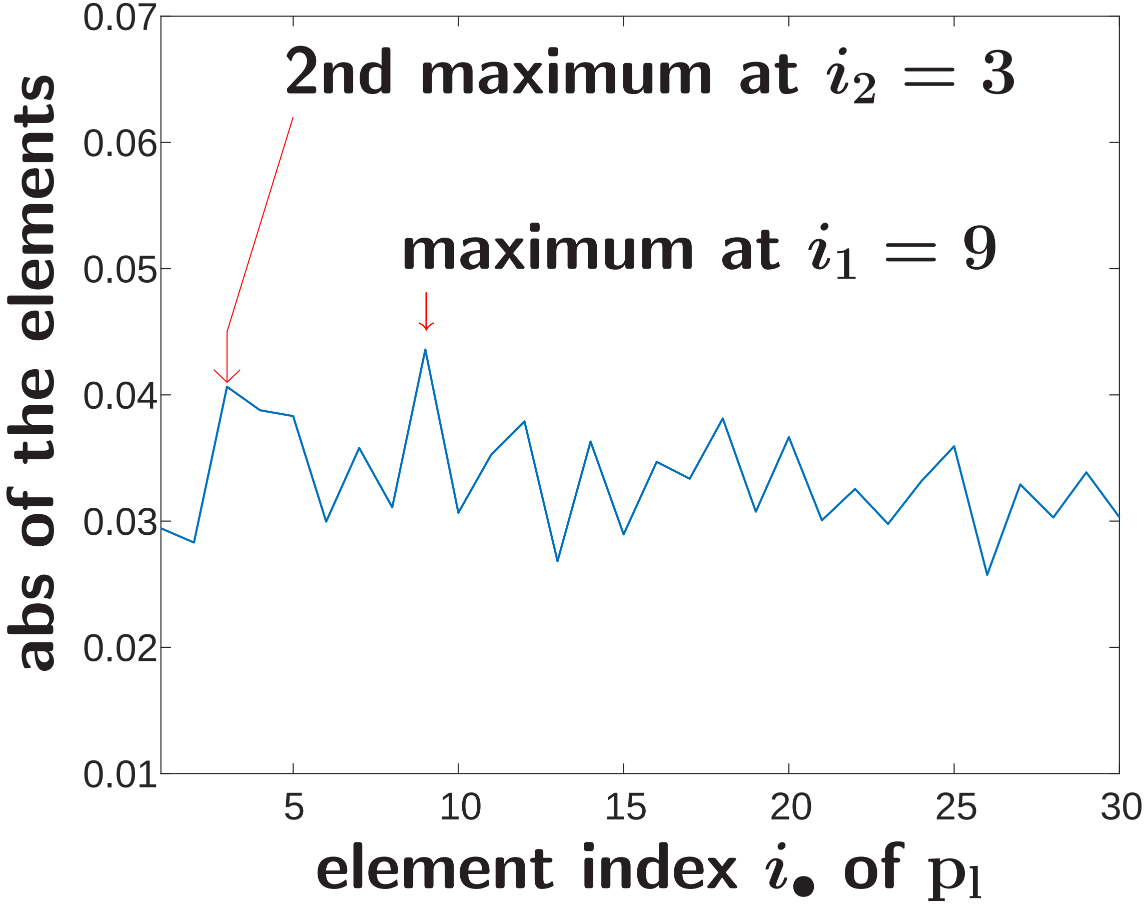

maximum at $i = 10$
0.04
0.035
0.03
0.025
0.02
5
10
15
20
25
30
element index i of $\mathrm{p_l}$
abs of the elements

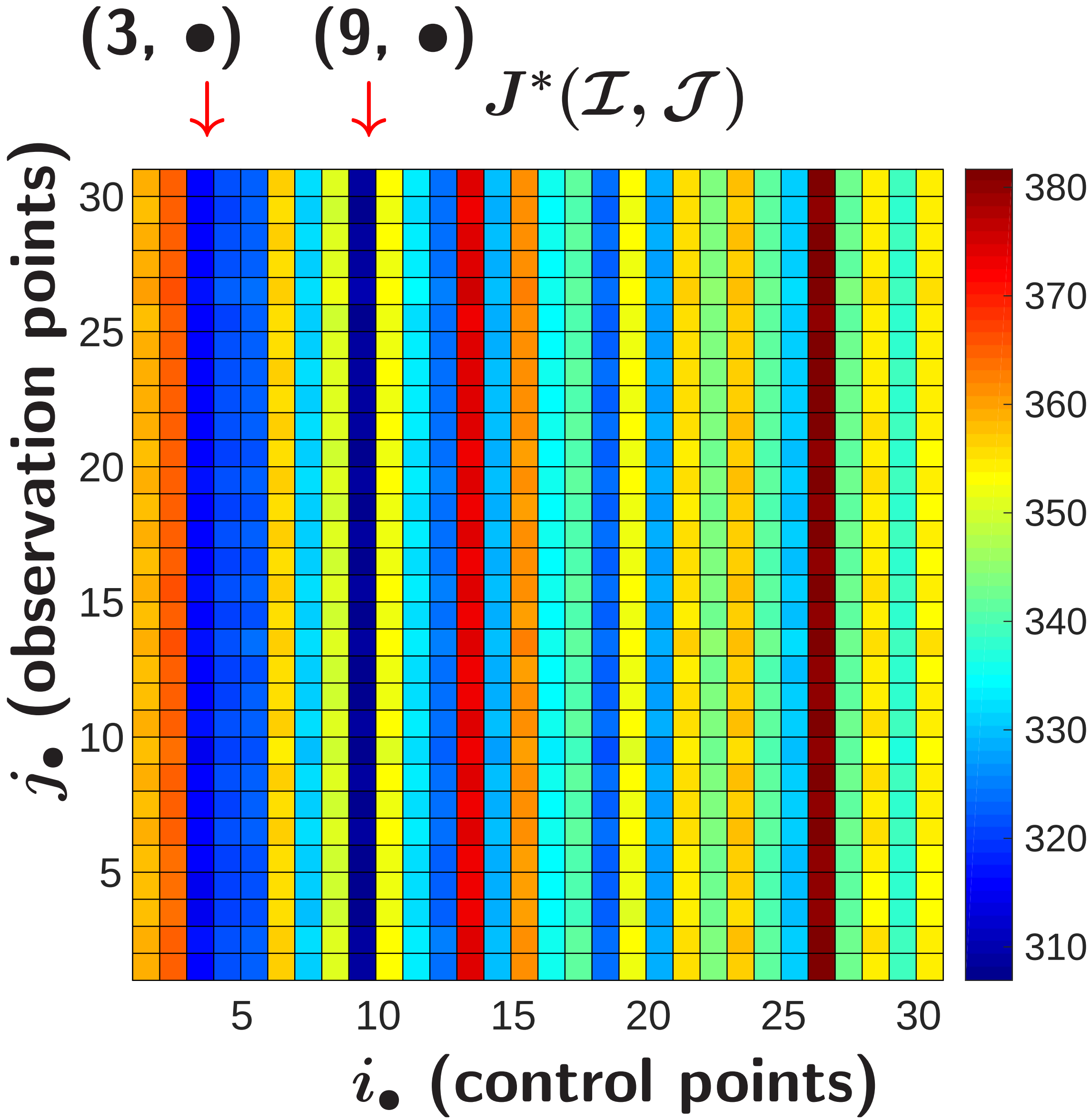

maximum at $j = 14$
0.05
0.045
0.04
0.035
0.03
0.025
0.02
abs of the elements
5
10
15
20
25
30
element index j of $\mathrm{p_r}$


$J^*(\mathcal{I}, \mathcal{J})$
(10, 14)
j (observation points)
i (control points)
5
10
15
20
25
30
260
280
300
320
340
360

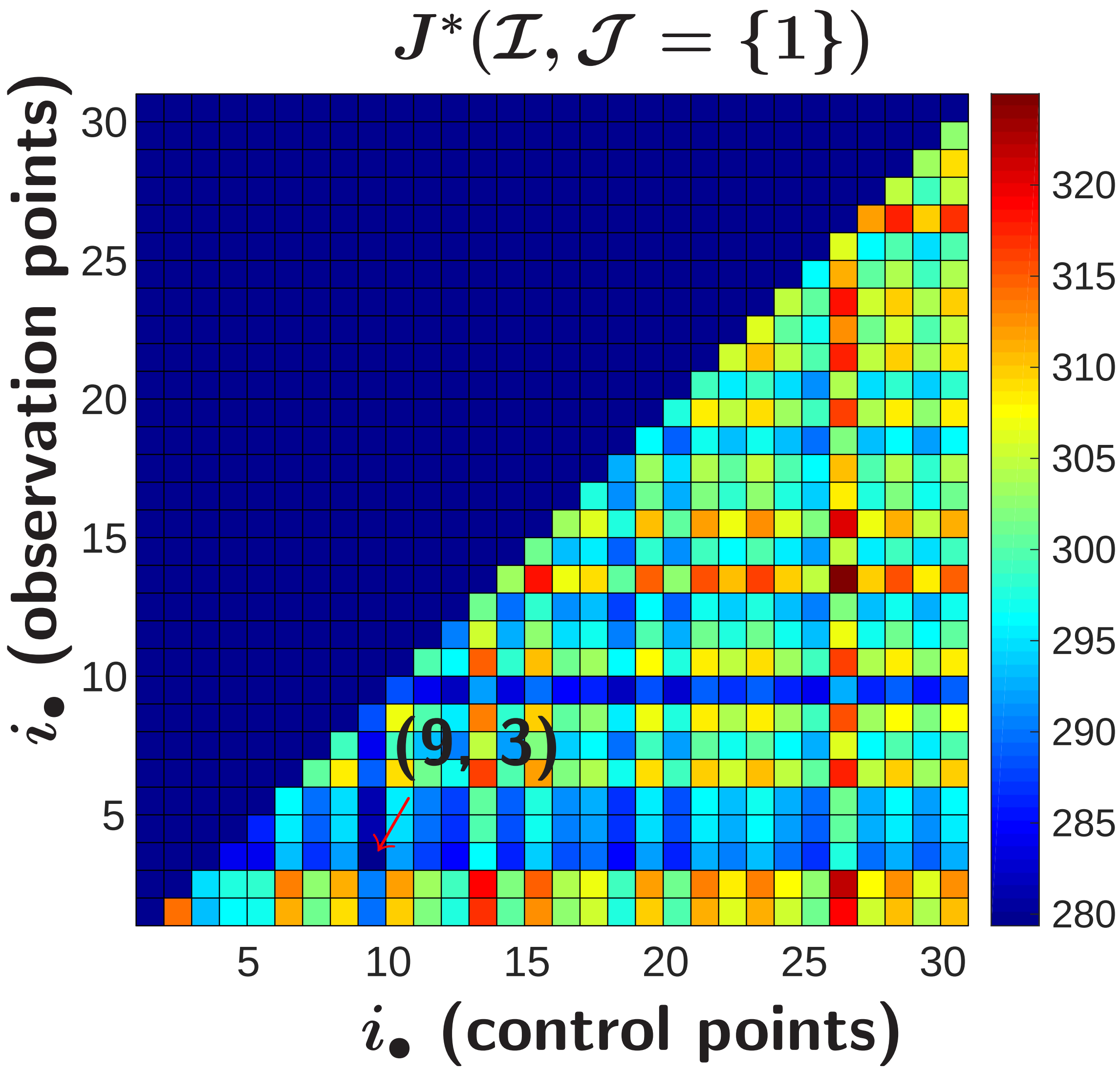
p_{qr} $p_{w_1 w_2}$	10	10^2	10^3	10^4
CR	**89%**	**96%**	**98%**	**100%**
MER	$1.83*10^{-3}$	$5.51*10^{-4}$	$7.99 *10^{-5}$	0

$A = A_{\mathrm{rs}}$: row stochastic matrix



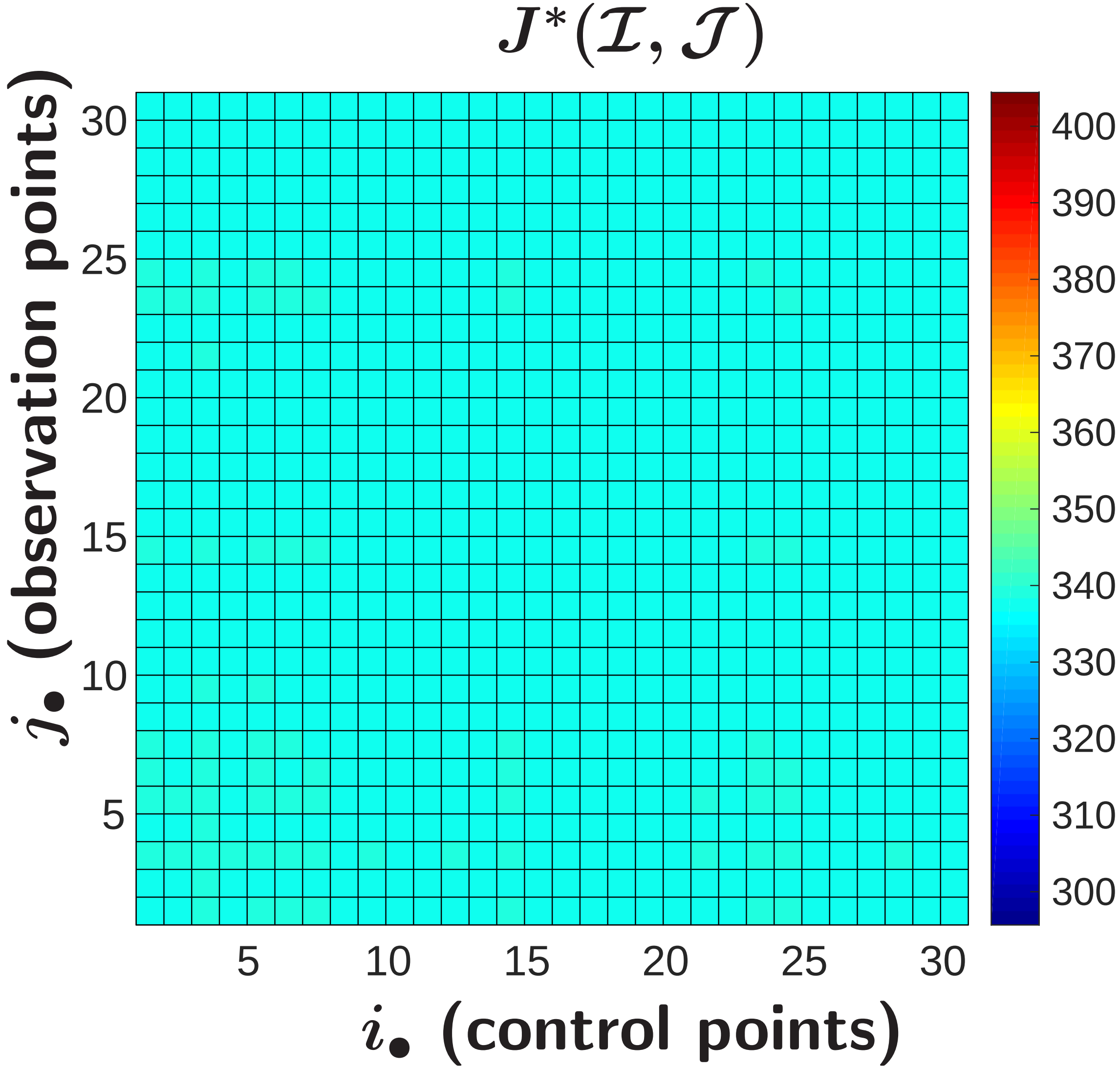


(3, •)
(9, •)
$J^*(\mathcal{I}, \mathcal{J})$
$j_\bullet$ (observation points)
$i_\bullet$ (control points)
5
10
15
20
25
30
310
320
330
340
350
360
370
380



$J^*(\mathcal{I}, \mathcal{J} = \{1\})$
30
25
20
15
10
5
$i_\bullet$ (observation points)
5
10
15
20
25
30
$i_\bullet$ (control points)
320
315
310
305
300
295
290
285
280
(9, 3)

$A = A_{\mathrm{ds}}$: doubly stochastic matrix



Recap

- **large-scaled multi-agent system**
- **optimal control/observation points problem**
- **left or right zero eigenvector of A**
- **drastically smaller computation number**