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Performance Competition in Cooperative Capturing by Multi-Agent Systems

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Abstract: In this paper, we deal with a problem to capture a target by linear multi-agent systems where the agents behave autonomously, whereas the target escapes with a reasonable strategy. We consider two cases for the dynamics of the agents and the target. First, for the simple dynamic case, we give a necessary and sufficient condition for the success of the capturing. Then, we extend the results to the general dynamics case and give similar sufficient conditions. The conditions clarify the performance competition between the target and the agents and we propose preferable strategies for them. We also demonstrate the results by using numerical simulations.

Key Words: multi-agent system, cooperative capturing, performance competition, Gershgorin theorem, transformed frequency variable

1. Introduction

In recent years, formation control composed of many agents such as a flock of robots, vehicles, aircrafts, artificial satellites or biological systems, has been actively investigated [1]–[15]. As a topic of it, cooperative pursuit of an object by a group of agents with their local information has been also discussed [5],[7],[8],[12],[13]. In particular, a control method of cyclic pursuit, which models the behavior of biological systems such as a flock of birds or fishes, was proposed in [7],[8],[13]. This kind of research is motivated by engineering senses for applications and also by scientific interests to clarify the behavior of biological systems. Examples of the applications are found in [7],[8],[12].

In the research area of pursuit or capturing, Kim and Sugie [16] proposed an effective cyclic pursuit of a target in random moving by a group of agents. They also demonstrated its efficiency by numerical simulations. However, a theoretical condition for the success of pursuit is not given and the dynamics of the agents and the target is considerably simple. On the other hand, Hara et al. [17] proposed a method to analyze the stability of formation control composed of agents and a target with general dynamics. They considered a transformation of the frequency variable of linear systems and gave a stability condition with the eigenvalues of an adjacent matrix which represents the interchange of information between the agents and the target.

The results in this research area of pursuit or capturing are interesting, however, the assumption that the position of the target is fixed or it moves regardless of the agents is artificial and it is more realistic to suppose the target takes reasonable behavior for escape. From this point of view, in this paper, we consider a problem of capturing a target by a group agents where the target is supposed to escape along a reasonable strategy. Then, we give conditions for the success of the capturing or the escape with respect to their control performance indices. In particular, we consider the problem in the following cases:

(i) simple dynamics of a target and agents,

(ii) general dynamics of a target and agents.

In (i), we deal with a case that the dynamics of the target and the agents is a simple linear 1st order system. Then, we give a necessary and sufficient condition for the capturing by employing the Gershgorin theorem with detailed analysis on the coefficients of the characteristic polynomial. In (ii), we consider more general dynamics of linear systems for the target and the agents. For this purpose, we employ a notion of a transformation of the frequency variable [17] and give conditions for the capturing with the eigenvalues of an adjacent matrix and the transformed stable/unstable region on the complex plane. Furthermore, we discuss the performance competition between the target and the agents for the capturing or the escape and propose preferable strategies of behavior for the target or the agents.

A relevant research is [4], in which a formation control by agents other than capturing is discussed. There is no target and each agent has its own cost function for constructing a formation. The objective is to find a compromised solution for all the agents by a game theoretic approach and a competition between a target and agents as in this paper is not discussed.

Note that, in this paper, our focus is on clarifying the essence of the competition in the capturing or the escape from the control theoretic viewpoint. Therefore, we intend to avoid detailed modeling of the realistic behavior of the target or the agents. We rather simplify the dynamics and try to derive clear conditions in this paper.

This paper is organized as follows. In Section 2, we prepare several notions and propositions used in the following of this paper, and formulate the cooperative capturing by multi-agent

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systems. We give a condition for capturing in Section 3 with a case of simple dynamics of the target and the agents. We also discuss the performance competition and propose preferable strategies for the target and the agents. In Section 4, we extend the result to the case of more general dynamics. Finally, we conclude the paper in Section 5.

Notation:

$\mathbb{R}$: real numbers, $\mathbb{R}_+$: nonnegative real numbers, $\mathbb{R}^n$: n-dimensional real vectors, $\mathbb{R}^{m\times n}$: $m \times n$-dimensional real matrices, $\mathbb{C}$: complex numbers, $\mathbb{C}_+$: the right half plane including the imaginary axis, $\mathbb{C}_+^c$: the complement of $\mathbb{C}_+$, A^{T}: the transpose of a matrix A, $\sigma_{n,r}$: a rth order elementary symmetric polynomial on $k_1, \ldots, k_n$.

2. Preliminary and Formulation

In this section, at first, we introduce a standard stability of linear systems and the related propositions used in this paper.

In general, for an nth order time invariant system:

$$\dot{x} = f(x), \quad t \geq 0, \quad x(0) = x_0, \tag{1}$$

x_e satisfying $f(x_e) = 0$ is called an *equilibrium point* of (1).

Definition 2.1 Let x_e be an equilibrium of (1). For arbitrary positive number $\epsilon > 0$, there exists a positive number $\delta(\epsilon) > 0$ and for any initial condition x_0 satisfying $\|x_0 - x_e\| \leq \delta(\epsilon)$,

$$\|x(t) - x_e\| \leq \epsilon, \quad \forall t \geq 0, \tag{2}$$

then, (1) is *stable* on x_e.

Proposition 2.1 (e.g., see [18]) For an nth order linear time invariant system:

$$\dot{x} = Ax, \quad t \geq 0, \quad x(0) = x_0, \tag{3}$$

the following statements are equivalent:

(i) the system (3) is stable (on $x_e = 0$).

(ii) the real part of the all eigenvalues of A are nonpositive and the eigenvalues of which the real parts are zero are simple roots of the minimal polynomial of A, if they exist.

Proposition 2.2 (Gershgorin Theorem [19]) For a complex matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, define discs on the complex plane as

$$\mathrm{D}_i := \left\{ s : |s - a_{ii}| \leq \sum_{j=1, j\neq i}^{n} |a_{ij}| \right\}, \; i = 1, 2, \ldots, n. \tag{4}$$

Then, the all eigenvalues of A are in the set of $\bigcup_{i=1}^{n} \mathrm{D}_i$.

We next explain the stability of systems with a particular structure:

$$\mathcal{G}(s) = C\left(\frac{1}{\nu(s)} I - A\right)^{-1} B + D \tag{5}$$

where $\nu(s)$ is supposed to be a continuous and strictly proper function of s. Let $\phi(s) := 1/\nu(s)$, then, we call a region which is transformed from $\mathbb{C}_+$ by $\phi(s)$ as Ω_+, its boundary $\partial\Omega_+$. The complement of Ω_+ is called Ω_+^c.

Proposition 2.3 (e.g., see [20]) Let $G(s)$ be a linear time invariant system. Then, the system $\mathcal{G}(s) = G(\phi(s))$ with a variable transformation $\phi(s)$ is stable iff the all poles of $G(s)$ except for single poles on $\partial\Omega_+$ are in Ω_+^c.

Hereafter, we formulate the problem of capturing. We consider a subsystem P_t, which tries to escape, called *target* and n subsystems $P_1, P_2, \ldots, P_n$, which try to capture the target, called *agents*. Their movements are supposed on a two dimensional x–y plane. We denote their coordinates by

$$p_i(t) = \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} \in \mathbb{R}^2, \; p_t(t) = \begin{bmatrix} x_t(t) \\ y_t(t) \end{bmatrix} \in \mathbb{R}^2, \tag{6}$$

and their dynamics is given by

$$\dot{p}_i = f_i(p_i, u_i), \tag{7}$$
$$\dot{p}_t = f_t(p_t, u_t), \tag{8}$$

where $f_i(\cdot)$ or $f_t(\cdot)$ is an appropriate linear function such as the equation of motion in Cartesian coordinate system, u_i and u_t are the control inputs of an agent and the target.

In this paper, we define capturing as follows:

Definition 2.2 For any given $\epsilon > 0$, if there exists a constant $\delta > 0$ and for any initial conditions $p_i(0)$, $p_t(0)$ such as $\|p_i(0) - \{p_t(0) + l_i\}\| < \delta$,

$$\|p_i(t) - \{p_t(t) + l_i\}\| < \epsilon, \;\; \forall t \tag{9}$$

is satisfied, then we call the *capturing is attained*.

The point $p_t(t) + l_i$ can be regarded as the object for the agent P_i where each parameter $l_i := [l_{x,i} \; l_{y,i}]^{\mathrm{T}}$ is given in advance in order to surround the target. When (9) is satisfied, the agents P_i keep the relative positions around the target and succeed to track the target P_t.

3. Cooperative Capturing: A Case of Simple Dynamics

3.1 State Space Representation

In this section, we deal with a case of simple dynamics (7) and (8) as:

$$\dot{p}_i = u_i, \tag{10}$$
$$\dot{p}_t = u_t. \tag{11}$$

On the other hand, the control law of the agents for capturing is given below.

Control law of the agents:

$$u_i = -k_i\{p_i - (p_t + l_i)\}, \tag{12}$$

where

$$k_i := \begin{bmatrix} k_{x,i} & 0 \\ 0 & k_{y,i} \end{bmatrix}, \; k_{x,i} > 0, \; k_{y,i} > 0.$$

For the agents (10) with (12), a large k_i implies that the agent is sensitive for the move of the target and in general, the objective (9) tends to be satisfied. In this sense, k_i can be regarded as the *performance index* of the agents.

Next, we define the escape strategy; the control law, of the target as follows.

Escape strategy of the target:

$$u_t = \sum_{i=1}^{n} \alpha_i k_t (p_t - p_i), \tag{13}$$

where

$$k_t := \begin{bmatrix} k_{x,t} & 0 \\ 0 & k_{y,t} \end{bmatrix}, \; k_{x,t} \geq 0, \; k_{y,t} \geq 0, \tag{14}$$

$$\alpha_i := \begin{bmatrix} \alpha_{x,i} & 0 \\ 0 & \alpha_{y,i} \end{bmatrix}, \; \alpha_{x,i} > 0, \; \alpha_{y,i} > 0,$$

$$\alpha_{x,1} + \alpha_{x,2} + \cdots + \alpha_{x,n} = 1,$$

$$\alpha_{y,1} + \alpha_{y,2} + \cdots + \alpha_{y,n} = 1.$$

The escape strategy (13) works such as there exist repulsions between the target and the agents, and k_t can be regarded as the performance index of the target. The weights α_i are tuning parameters for the target. A large α_i implies that the target strongly escapes from the agent P_i. In this sense, the weights α_i assign the distribution ratios of the total performance k_t against each agent P_i.

Remark 3.1 In general, the capturing behavior of realistic biological systems is composed of many complex elements, therefore, the problem formulation given above is a considerably simplified model. However, it describes a fundamental principle of capturing or escape and the results in the following of this paper clarify the performance competition in the cooperative capturing and escape.

Remark 3.2 On the control inputs (12) and (13), we assume that each agent can use the relative distance between its location and the target's location and on the other hand, the target can use the relative distances between its location and the all agents.

As seen from (13), the dynamics on the axis x and y are independent each other, therefore, in order to avoid the redundancy on the discussion and the notations, in the following of this paper, we basically refer to the x axis only and omit the subscript x in the notations (e.g., $k_{x,i} \to k_i$, $l_{x,i} \to l_i$, $\alpha_{x,i} \to \alpha_i$).

We next prepare the augmented system of the whole systems (10)–(13) in the following state space representation (pick up x-axis only),

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{l}, \\ \boldsymbol{x} &= \begin{bmatrix} x_t & x_1 & x_2 & \cdots & x_n \end{bmatrix}^{\mathrm{T}}, \\ \boldsymbol{l} &= \begin{bmatrix} 0 & l_1 & l_2 & \cdots & l_n \end{bmatrix}^{\mathrm{T}}, \end{aligned} \tag{15}$$

where

$$A := \left[\begin{array}{c|cccc} k_t & -\alpha_1 k_t & -\alpha_2 k_t & \cdots & -\alpha_n k_t \\ \hline k_1 & -k_1 & & & O \\ k_2 & & -k_2 & & \\ \vdots & & & \ddots & \\ k_n & O & & & -k_n \end{array}\right], \tag{16}$$

$$B := \left[\begin{array}{c|cccc} k_t & 0 & 0 & \cdots & 0 \\ \hline 0 & k_1 & & & O \\ 0 & & k_2 & & \\ \vdots & & & \ddots & \\ 0 & O & & & k_n \end{array}\right].$$

The block diagram of the whole system is given in Fig. 1.

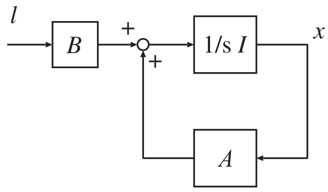

Fig. 1 The block diagram of system (15): a case of simple dynamics.

3.2 A Condition for Capturing: A Case of Simple Dynamics

We get the following result:

Theorem 3.1 In the system (15) composed of a target P_t and n agents P_i, a necessary and sufficient condition for the capturing is

$$k_t \sum_{i=1}^{n} \frac{\alpha_i}{k_i} < 1. \tag{17}$$

The proof is given in Section 3.4.

The condition (17) gives us insight on performance competition between the agents and the target and the preferable strategies for them are explained as follows:

Performance as a group

The condition (17) can be deformed into

$$k_t < \left(\sum_{i=1}^{n} \frac{\alpha_i}{k_i} \right)^{-1}. \tag{18}$$

The right hand side is the weighted harmonic mean of k_i and it represents the performance of the agents as a whole.

The strategy for the agents

In general, according to the relationship between the weighted harmonic mean and the arithmetical mean, $\left(\sum_{i=1}^{n} \frac{\alpha_i}{k_i}\right)^{-1}$ is bounded by

$$\left(\sum_{i=1}^{n} \frac{\alpha_i}{k_i} \right)^{-1} \leq \sum_{i=1}^{n} \alpha_i k_i,$$

and the equality is held at $k_1 = k_2 = \cdots = k_n =: k$. Therefore, when α_i are fixed and $\sum_{i=1}^{n} \alpha_i k_i$ is constant, $k_1 = k_2 = \cdots = k_n = k$ is the best strategy for the agents regardless of α_i. This implies the agents should always form a "homogeneous" group whether they know the control strategy α_i of the target or not. Moreover, when $k_1 = k_2 = \cdots = k_n = k$, (17) or (18) becomes

$$k_t < k \tag{19}$$

and the condition for the capture or the escape is a simple comparison between their individual performances k_t and k.

The strategy for the target

Suppose that α_i is a tuning parameter for the target. When the agents do not choose the strategy mentioned above and the target knows the order $k_1 > k_2 > \cdots > k_n$, then $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ is a preferable setting of the weights for the target in order to decrease the right hand side of (18) under the limited control input. This implies that setting a large weight α_i corresponding to the small k_i, i.e. a weak agent, and strong escape from it, is a preferable strategy for the target. On the other hand, when the agents choose the strategy $k_1 = k_2 = \cdots = k_n = k$, then there is no difference of the choice α_i for decreasing the right hand side of (18) and (18) results in (19).

Remark 3.3 Theorem 3.1 states only a fundamental fact on the competition of the feedback gains and we intend to show that the result has many usages such as discussed after Theorem 3.1. Some cases of the assumptions and the resultant conditions for capturing are summarized as follows:

(i) Suppose that the agents do not necessarily choose the best strategy and both of the agents and the target do not know the parameter settings of the opponents each other, then the condition for capturing is (18).

(ii) Suppose that the agents do not necessarily choose the best strategy and only the target knows the parameter setting of the agents $k_1 > k_2 > \cdots > k_n$, then the condition for capturing is (18) and a preferable setting of α_i is $\alpha_1 < \alpha_2 < \cdots < \alpha_n$.

(iii) Suppose that the agents choose the best strategy $k_i = k$, $\forall i$ under a condition of a constant $\sum_{i=1}^{n} \alpha_i k_i$, then the condition for capturing results in (19) whether the target knows this strategy of the opponent or not.

More exact game theoretic discussion is beyond the current objective of this paper and left for future work.

3.3 Numerical Simulation I

We show numerical simulations to demonstrate the results. Let $n = 4$ and

$$\begin{aligned} k_1 &= \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}, \; k_2 = \begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix} \\ k_3 &= \begin{bmatrix} 3.5 & 0 \\ 0 & 3.5 \end{bmatrix}, \; k_4 = \begin{bmatrix} 4.5 & 0 \\ 0 & 4.5 \end{bmatrix} \\ l_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; l_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ l_3 &= \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \; l_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ p_1(0) &= \begin{bmatrix} 7 \\ 7 \end{bmatrix}, \; p_2(0) = \begin{bmatrix} -7 \\ 7 \end{bmatrix} \\ p_3(0) &= \begin{bmatrix} -7 \\ -7 \end{bmatrix}, \; p_4(0) = \begin{bmatrix} 7 \\ -7 \end{bmatrix}. \end{aligned} \tag{20}$$

We also set k_t and the weights α_i as

$$k_t = \begin{bmatrix} 2.0 & 0 \\ 0 & 2.0 \end{bmatrix}, \tag{21}$$

$$\begin{aligned} \alpha_1 &= \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \; \alpha_2 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, \\ \alpha_3 &= \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \; \alpha_4 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}. \end{aligned} \tag{22}$$

Note that the weights are normalized as $\sum \alpha_i = I$. Then,

$$k_{x,t} \sum_{i=1}^{4} \frac{\alpha_{x,i}}{k_{x,i}} = k_{y,t} \sum_{i=1}^{4} \frac{\alpha_{y,i}}{k_{y,i}} = 0.7343 < 1, \tag{23}$$

therefore, the condition (17) for capturing is attained on x-axis and y-axis, simultaneously. Figure 2 shows the loci of the agents P_i ($i = 1, 2, 3, 4$) and the target P_t, and it is known that actually the capturing is attained.

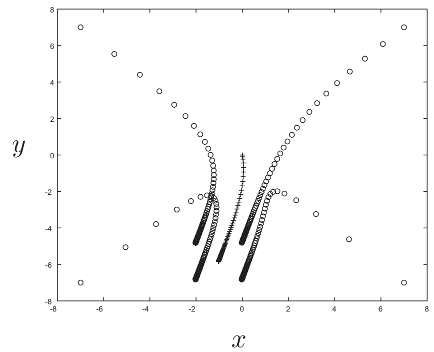

Fig. 2 The loci of the agents P_i ($i = 1, 2, 3, 4$, marked by '∘') and the target P_t (marked by '+') (a case of capture).

Next, since the performance k_1 of the agent P_1 is the weakest among the agents, tune the weights α_i by

$$\begin{aligned} \alpha_1 &= \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \; \alpha_2 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, \\ \alpha_3 &= \begin{bmatrix} 0.15 & 0 \\ 0 & 0.15 \end{bmatrix}, \; \alpha_4 = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix}, \end{aligned} \tag{24}$$

whereas keep $\sum \alpha_i = I$. This means the target escapes from the weakest agent P_1 strongly. In this case,

$$k_{x,t} \sum_{i=1}^{4} \frac{\alpha_{x,i}}{k_{x,i}} = k_{y,t} \sum_{i=1}^{4} \frac{\alpha_{y,i}}{k_{y,i}} = 1.0146 > 1 \tag{25}$$

and the condition (17) is not satisfied. Actually, Fig. 3 shows that the capturing fails and the target succeeds to escape.

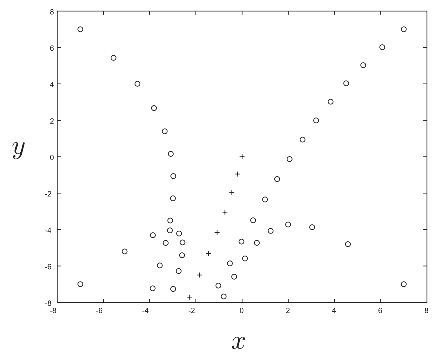

Fig. 3 The loci of the agents P_i ($i = 1, 2, 3, 4$, marked by '∘') and the target P_t (marked by '+') (a case of escape).

The above two cases demonstrate that the tuning of the weights α_i is important for the target to escape.

3.4 Proof of Theorem 3.1

We check the location of the eigenvalues of A with respect to $k_t(\geq 0)$ for a fixed $k_i(> 0)$.

From the Gershgorin Theorem (Proposition 2.2), the all eigenvalues of A are located in the region of the union of a disc centered at $(k_t, 0)$ with radius k_t (call D_t hereafter) and discs centered at $(-k_i, 0)$ with radius k_i ($i = 1, 2, \cdots, n$) (call D_i hereafter) in the complex plane. This implies that the all discs touch the imaginary axis only at the origin for any k_t and k_i, and the discs D_i exist in the left half plane for any $k_i > 0$. When $k_t = 0$, the disc D_t becomes a point located at the origin, therefore, the all discs exist in the closed left half plane. Next, when $k_t > 0$, only the disc D_t extends in the right half plane which still touch the imaginary axis only at the origin (see Fig. 4).

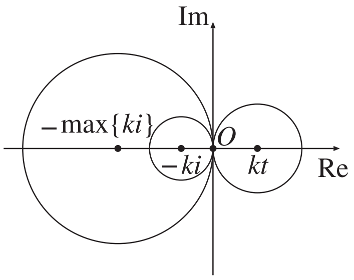

Fig. 4 The location of the Gershgorin discs in the case $k_t > 0$ and $k_i > 0$.

From the above facts, we show the condition (17) for the stability with the following steps:

(i) The eigenvalues of A are continuous functions of k_t (their loci on the complex plane with respect to k_t are continuous).

(ii) When the eigenvalues move from the left half plane to the right half plane (and vice versa) by the continuous change of k_t, they necessarily pass through the origin.

(iii) The matrix A always has an eigenvalue at the origin.

(iv) The other eigenvalues should be asymptotically stable for (9).

(v) When $k_t = 0$, the other eigenvalues exist in the open left half plane, that is, capturing is attained.

(vi) The value of k_t which satisfies the condition that one of the other eigenvalues referred in (v) is located on the origin is unique (call $\bar{k}_t$). Moreover, when $0 \le k_t < \bar{k}_t$, the eigenvalues referred in (v) are asymptotically stable, otherwise they are marginally stable or unstable.

The statement (i) is obvious from the definition of A. With (i), the statement (ii) is also obvious since the Gershgorin discs D_t and D_i touch the imaginary axis only at the origin from Proposition 2.2.

Next, by using a nonsingular matrix

$$T = \left[\begin{array}{c|cccc} 1 & 0 & 0 & \cdots & 0 \\ \hline 1 & 1 & & & O \\ 1 & & 1 & & \\ \vdots & & & \ddots & \\ 1 & O & & & 1 \end{array}\right],$$

transform the state variable x in (15) as

$$\begin{aligned} \boldsymbol{x} &= T\boldsymbol{z}, \\ \boldsymbol{z} &:= \left[\begin{array}{ccccc} z_t & z_1 & z_2 & \cdots & z_n \end{array}\right]^{\mathrm{T}}, \end{aligned} \tag{26}$$

then,

$$\dot{\boldsymbol{z}} = T^{-1}AT\boldsymbol{z} + T^{-1}B\boldsymbol{l} =: \tilde{A}\boldsymbol{z} + T^{-1}B\boldsymbol{l},$$

where $\tilde{A}$ is given by

$$\begin{aligned} \tilde{A} &= T^{-1}AT \\ &= \left[\begin{array}{c|cccc} 0 & -w_1 & -w_2 & \cdots & -w_n \\ \hline 0 & -k_1 + w_1 & w_2 & \cdots & w_n \\ 0 & w_1 & -k_2 + w_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & w_n \\ 0 & w_1 & \cdots & w_{n-1} & -k_n + w_n \end{array}\right], \end{aligned} \tag{27}$$

and

$$w_i := \alpha_i k_t, \quad i = 1, 2, \cdots, n. \tag{28}$$

From this, (iii) is given. The statement (iv) is from (iii) and Proposition 2.1.

Next, denote the 2-2 block matrix of $\tilde{A}$ as

$$\tilde{A}_{22} = \left[\begin{array}{cccc} -k_1 + w_1 & w_2 & \cdots & w_n \\ w_1 & -k_2 + w_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & w_n \\ w_1 & \cdots & w_{n-1} & -k_n + w_n \end{array}\right]. \tag{29}$$

Let $\psi(s)$ be the characteristic polynomial of $\tilde{A}_{22}$, then a direct calculation gives

$$\begin{aligned} \psi(s) &= \det(sI - \tilde{A}_{22}) \\ &= \prod_{i=1}^{n}(s + k_i) - \sum_{i=1}^{n}\left(w_i \prod_{j=1, j\neq i}^{n}(s + k_j)\right) \\ &=: s^n + c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \cdots + c_1 s + c_0, \end{aligned} \tag{30}$$

where the coefficient c_0 is described by

$$\begin{aligned} c_0 =& \sigma_{n,n} - \sum_{i=1}^{n}(\sigma_{n,n-1} - k_i\sigma_{n,n-2} + k_i^2\sigma_{n,n-3} - \\ & \cdots + (-1)^{n-1}k_i^{n-1})w_i \end{aligned}$$

with elementary symmetric polynomials $\sigma_{n,j}$ on k_1–k_n defined in Definition A.1. From the definition of $\sigma_{n,j}$, it is obvious that $\sigma_{n,j} > 0$ for all j.

When $k_t = 0$,

$$\psi(0) = \sigma_{n,n} > 0. \tag{31}$$

This implies $\tilde{A}_{22}$ does not have an eigenvalue at origin when $k_t = 0$. However, recall that when $k_t = 0$, the all Gershgorin discs exist in the close left half plane and they only touch the imaginary axis at the origin. Therefore, (v) is concluded.

Finally, we show (vi). At first, note that the signatures of w_i and k_t are same for all i from the definition of w_i. On the other hand, we can also show that

$$\sigma_{n,p-1} - k_i\sigma_{n,p-2} + k_i^2\sigma_{n,p-3} - \cdots + (-1)^{p-1}k_i^{p-1} > 0$$

for any p and i by Lemma A.1. This implies c_0 is a uniformly decreasing number with respect to k_t for fixed k_i. Therefore,

$$\bar{k}_t \;\; \text{s.t.} \;\; \psi(0) = c_0 = 0 \tag{32}$$

is unique. From the fact (i), (ii) and (v), when $0 \le k_t < \bar{k}_t$, $\psi(0) = c_0 > 0$ and $\tilde{A}_{22}$ is asymptotically stable. Contrary, when

$k_t \geq \bar{k}_t$, $\psi(0) = c_0 \leq 0$ and $\tilde{A}_{22}$ is unstable or marginaly stable. This concludes (vi).

From (iii), (iv) and (vi), $0 \leq k_t < \bar{k}_t$ or equivalently $\psi(0) = c_0 > 0$ is the necessary and sufficient condition for the capturing.

Finally, we describe the condition

$$\psi(0) = c_0 > 0 \tag{33}$$

by using k_t, k_i and α_i. A direct calculation gives:

$$(33) \Longleftrightarrow \sigma_{n,n} - \sum_{i=1}^{n}(\sigma_{n,n-1} - k_i\sigma_{n,n-2} + k_i^2\sigma_{n,n-3} - \cdots + (-1)^{n-1}k_i^{n-1})w_i > 0 \tag{34}$$

$$\Longleftrightarrow \prod_{i=1}^{n} k_i - w_1\prod_{i=2}^{n} k_i - w_2\prod_{\substack{i=1\\i\neq 2}}^{n} k_i - w_3\prod_{\substack{i=1\\i\neq 3}}^{n} k_i - \cdots - w_n\prod_{i=1}^{n-1} k_i > 0 \tag{35}$$

$$\Longleftrightarrow w_1\prod_{i=2}^{n} k_i + w_2\prod_{\substack{i=1\\i\neq 2}}^{n} k_i + w_3\prod_{\substack{i=1\\i\neq 3}}^{n} k_i + \cdots + w_n\prod_{i=1}^{n-1} k_i < \prod_{i=1}^{n} k_i \tag{36}$$

$$\Longleftrightarrow \sum_{i=1}^{n}\frac{w_i}{k_i} = k_t\sum_{i=1}^{n}\frac{\alpha_i}{k_i} < 1. \tag{37}$$

This concludes the statement of the theorem. □

4. Cooperative Capturing: The Case of General Dynamics

In Section 3, we deal with a simple case of the dynamics of the target and the agents. In order to correspond to more realistic situations, we extend the class in this section. Note that we consider the case:

$$\alpha_i = \frac{1}{n},\ \forall i, \tag{38}$$

in this section for simplifying the problem.

4.1 State Space Representation with a Transformed Frequency Variable

In this section, we consider the following dynamics for the agents and the target.

$$p_i = \begin{bmatrix} v(s) & 0 \\ 0 & v(s) \end{bmatrix} u_i \tag{39}$$

$$p_t = \begin{bmatrix} v(s) & 0 \\ 0 & v(s) \end{bmatrix} u_t \tag{40}$$

The control inputs of the agents and the target are the same of (12) and (13), respectively.

We also describe the dynamics on x-axis only such as in Section 3, then we get the following equation corresponding to (15):

$$\begin{aligned} \frac{1}{v(s)}\boldsymbol{x} &= A\boldsymbol{x} + B\boldsymbol{l}, \\ \boldsymbol{x} &= \begin{bmatrix} x_t & x_1 & x_2 & \cdots & x_n \end{bmatrix}^{\mathrm{T}}, \\ \boldsymbol{l} &= \begin{bmatrix} 0 & l_1 & l_2 & \cdots & l_n \end{bmatrix}^{\mathrm{T}}, \end{aligned} \tag{41}$$

where A is defined in (16) with $\alpha_i = \frac{1}{n}$, $\forall i$. The corresponding block diagram is given in Fig. 5.

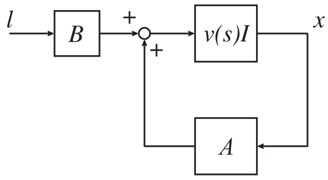

Fig. 5 The block diagram of system (41): the case of general dynamics.

The transfer function from $\boldsymbol{l}$ to $\boldsymbol{x}$ is given by

$$\mathcal{G}(s) = \left(\frac{1}{v(s)}I - A\right)^{-1} B. \tag{42}$$

By using the stability analysis explained in Section 2, (42) can be regarded as a linear system with a transformed frequency variable $\phi(s) := 1/v(s)$. In the following, we give conditions for capturing when $v(s)$ is in some classes defined below:

Definition 4.1 Denote the set of $\phi(s)$ which satisfies that Ω_+ $(:= \phi(\mathbb{C}_+))$ does not contain $\phi \in (-\mu, 0]$, $\mu > 0$ on the real axis by Φ_μ. Also define

$$\mathcal{V}_\mu := \left\{ v(s) = \frac{1}{\phi(s)},\ \phi(s) \in \Phi_\mu, v(s) : \text{strictly proper} \right\}. \tag{43}$$

For example,

$$v(s) = \frac{1}{s(s+a)},\ a > 0 \tag{44}$$

is in $\mathcal{V}_\infty$. This is the typical case that the dynamics of the agents and the target is a second order system with a mass point and a damper. The complement region Ω_+^c of Ω_+ when $a = 1.5$ is described as the shaded area in Fig. 6.

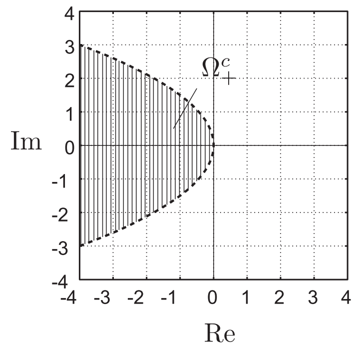

Fig. 6 The region Ω_+^c in the case $v(s) = \dfrac{1}{s(s+1.5)}$.

The boundary of the region Ω_+^c in Fig. 6 is given by

$$\phi(j\omega) = j\omega(j\omega + a) = -\omega^2 + j\omega a \tag{45}$$

and the set Ω_+^c can be regarded as a transformed stable region for the eigenvalues of A in (42). It is also shown that Ω_+ does not contain $(-\infty, 0]$ on the real axis.

On the other hand,

$$v(s) = \frac{1}{s}\cdot\frac{1}{ms^2 + ds + \lambda},\ m,\ d,\ \lambda > 0 \tag{46}$$

is in $\mathcal{V}_\mu$, where $0 < \mu < \infty$. This is another typical case of the dynamics with an integrater, a mass point, a spring and a damper. In this case, the complement region Ω_+^c of Ω_+ at $m = 1$, $d = 1.2$, $\lambda = 2.2$ is described as the shaded area in Fig. 7. It is also shown that Ω_+ does not contain $(-\mu, 0] \simeq (-2.7, 0]$ on the real axis.

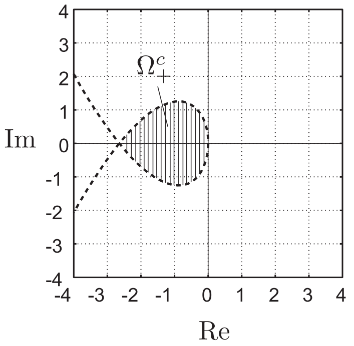

Fig. 7 The region Ω_+^c in the case $v(s) = \dfrac{1}{s(s^2 + 1.2s + 2.2)}$.

Remark 4.1 As in the above examples, μ can be easily calculated in a simple case of $v(s)$, i.e., $\phi(j\omega)$ gets across the real axis once or not. In general cases such as $\phi(j\omega)$ acrosses the real axis several times, it is not trivial to find μ, however a combination of several matrix inequalities can give μ numerically by employing a method in [21].

4.2 A Condition for Capturing: The Case of General Dynamics

Now, we give a condition for capturing:

Theorem 4.1 Assume

$$v(s) \in \mathcal{V}_\mu, \tag{47}$$

then, the following hold:

(i) When $0 < \mu < \infty$, a sufficient condition that the system (41) attains capturing is

$$\frac{k_t}{n} \sum_{i=1}^{n} \frac{1}{k_i} < 1 \tag{48}$$

and

$$-\mu < -2k_{\max}, \tag{49}$$

where $k_{\max} := \max\{k_1,\ k_2,\ \cdots,\ k_n\}$.

(ii) When $\mu = \infty$, a necessary and sufficient condition that the system (41) attains capturing is (48).

Proof Proof of (i): From Proposition 2.3, a condition that the all eigenvalues except for a zero eigenvalue of A are in Ω_+^c and the system attains capturing (9) are equivalent. On the other hand, Theorem 3.1 gives a necessary and sufficient condition (17) for that the all eigenvalues except for a zero eigenvalue of A are in the open left half plane. Note that when $\alpha_i = \frac{1}{n}$, $\forall i$, the matrix $\tilde{A}_{22}$ given in (29) is symmetry and the all eigenvalues of A are real. Therefore, their minimum is larger than or equal to $-2k_{\max}$ from the Gershgorin theorem. On the other hand, the unstable region Ω_+ for $v(s) \in \mathcal{V}_\mu$ does not contain $\phi \in (-\mu,\ 0]$ on the real axis. Therefore, (48) and (49) imply the all eigenvalues except a zero eigenvalue of A are in the stable region Ω_+^c. This concludes the statement.

Proof of (ii): In the case of $v(s) \in \mathcal{V}_\infty$, the unstable region Ω_+ on the complex plane does not contain the negative part of the real axis, therefore, the stability of the system (41) is equivalent to (48). □

From Theorem 4.1 and 3.1, it is known that the capturing conditions are identical for the first order system (10), (11) and for the system (39) and (40) where $v(s) \in \mathcal{V}_\infty$.

4.3 Numerical Simulation II

We show numerical simulations in a case of (44) with $a = 0.9$, that is, $f(s) \in \mathcal{V}_\infty$, (20) and (21). Note that the weights are

$$\alpha_i = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix},\ \forall i. \tag{50}$$

In this case, the necessary and sufficient condition (48) for capturing in Theorem 4.1-(ii) is satisfied as

$$\frac{k_{x,t}}{4} \sum_{i=1}^{4} \frac{1}{k_{x,i}} = \frac{k_{y,t}}{4} \sum_{i=1}^{4} \frac{1}{k_{y,i}} = 0.7873 < 1. \tag{51}$$

Figure 8 shows the loci of the agents P_i ($i = 1, 2, 3, 4$) and the target P_t. From the simulation, it is known that the capturing succeeds.

Next, we set k_t as

$$k_t = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \tag{52}$$

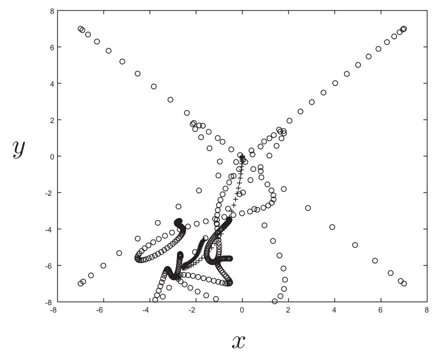

Fig. 8 The loci of the agents P_i ($i = 1, 2, 3, 4$, marked by '○') and the target P_t (marked by '+') (a case of $v(s) \in \mathcal{V}_\infty$ and success of capture).

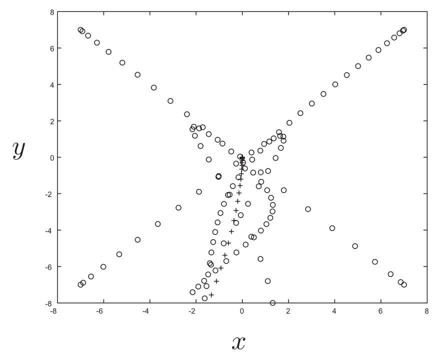

Fig. 9 The loci of the agents P_i ($i = 1, 2, 3, 4$, marked by '○') and the target P_t (marked by '+') (a case of $v(s) \in \mathcal{V}_\infty$ and escape).

then,

$$\frac{k_{x,t}}{4}\sum_{i=1}^{4}\frac{1}{k_{x,i}} = \frac{k_{y,t}}{4}\sum_{i=1}^{4}\frac{1}{k_{y,i}} = 1.1810 > 1. \tag{53}$$

This implies (48) is not satisfied. Figure 9 shows the numerical simulation of the loci of the agents and the target in this case. From the figure, it is known that the capturing fails.

5. Conclusion

In this paper, we considered a cooperative capturing problem by multi-agent systems where the target escapes with a reasonable strategy. We gave necessary and sufficient conditions or a sufficient condition for the cases of simple and general dynamics of the target and the agents. We furthermore discussed the meaning of the condition with respect to the performance competition between the target and the agents. In particular, we explained that a reasonable strategy for the target is to escape strongly from weak agents, and on the contrary, for the agents it is reasonable to form a homogeneous group.

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Appendix

We give the definition of elementary symmetric polynomials and the related lemmas.

Definition A.1 (elementary symmetric polynomial) For given $k_1, k_2, \ldots, k_n$, consider the expansion of $\prod_{i=1}^{n}(x+k_i)$ as

$$\begin{aligned}&\prod_{i=1}^{n}(x+k_i)\\&= x^n + \sigma_{n,1}x^{n-1} + \sigma_{n,2}x^{n-2} + \cdots + \sigma_{n,n-1}x + \sigma_{n,n}.\end{aligned} \tag{A. 1}$$

Then, $\sigma_{n,r}$ is called an elementary symmetric polynomial on k_1, $k_2, \ldots, k_n$.

For example, when $n = 3$,

$$\begin{aligned}\sigma_{3,1} &= k_1 + k_2 + k_3,\\ \sigma_{3,2} &= k_1k_2 + k_2k_3 + k_3k_1,\\ \sigma_{3,3} &= k_1k_2k_3.\end{aligned}$$

Lemma A.1 For arbitrary $p = 1, \ldots, n$ and $i = 1, \ldots, n$,

$$\sigma_{n,p-1} - k_i\sigma_{n,p-2} + k_i^2\sigma_{n,p-3} - \cdots + (-1)^{p-1}k_i^{p-1} > 0. \tag{A. 2}$$

Proof At first, the left hand side of the inequality can be deformed to

$$\begin{aligned}&\sigma_{n,p-1} - k_i\sigma_{n,p-2} + k_i^2\sigma_{n,p-3} - \cdots + (-1)^{p-1}k_i^{p-1}\\ &=\sigma_{n,p-1} - k_i(\sigma_{n,p-2} - k_i(\sigma_{n,p-3} -\\ &\quad \cdots - k_i(\sigma_{n,2} - k_i(\sigma_{n,1} - k_i))\cdots)).\end{aligned}$$

The most inside term of the brackets is positive as

$$\begin{aligned}&\sigma_{n,1} - k_i \\ &= k_1 + k_2 + \cdots + k_{i-1} + k_{i+1} + \cdots + k_n \\ &> 0 \quad (\because\ k_i > 0,\ \forall i).\end{aligned}$$

Next,

$$\begin{aligned}&\sigma_{n,2} - k_i(\sigma_{n,1} - k_i) \\ = \quad & k_1k_2 + k_1k_3 + \cdots k_1k_{i-1} + k_1k_{i+1} + \cdots k_1k_n \\ & + k_2k_3 + k_2k_4 + \cdots k_2k_{i-1} + k_2k_{i+1} + \cdots k_2k_n \\ & \vdots \\ >&0.\end{aligned}$$

Repeat this and get

$$\begin{aligned}&\sigma_{n,p-1} - k_i(\sigma_{n,p-2} - k_i(\sigma_{n,p-3} - \\ &\cdots - k_i(\sigma_{n,2} - k_i(\sigma_{n,1} - k_i))\cdots)) \\ =&\sigma_{n,p-1} \text{ except for the terms which contains } k_i \\ >&0.\end{aligned}$$

This concludes the statement. □

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