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Distributed Feedback Control of Quantum Networks ⋆

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Abstract: In this paper, we deal with a distributed feedback control of quantum networks called a quantum consensus algorithm (QCA) with local quantum observation and feedback proposed by Kamon & Ohki (2013, 2014) and prove strictly that QCA makes quantum states converge to a quantum state called symmetric state consensus (SSC) with probability one from arbitrary initial states keeping purity. The difficulty of the proof is from that the objective system is stochastic and non-linear, and we solve it by employing the stochastic Lyapunov stability analysis. We also show that QCA can generate a desirable W-state, which is known as an important entangled quantum state and utilized for many applications of quantum information technology.

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1. INTRODUCTION

Quantum control has been actively investigated to overcome such problems as the generation or preservation of quantum bits (qubits) under noisy environments Wiseman & Milburn (2009). From the establishment of quantum filtering theory Belavkin (1992), research about quantum control has advanced and contributed to broad areas of quantum information technologies Mirrahimi & van Handel (2007). However, as is the case with classical systems, it is quite difficult to control quantum bits when the number of bits is large because of the increasing complexity of instrument networks (e.g., see Yokoyama et al. (2013) for the case of optical systems). Then, a distributed operation called quantum consensus, which is one of the distributed quantum information applications, is a promising idea to generate quantum states of large-scale quantum systems.

Mazzarella et al. (2013) have developed a framework of quantum consensus as the extension of classical consensus problems. They have defined several types of quantum consensus states, derived their hierarchical relationship and proposed a quantum version of gossip algorithm which asymptotically generates a consensus state called symmetric state consensus (SSC).

Their algorithm can be regarded as an autonomous system like classical consensus systems and it contains no feedback input operation depending on the current quantum states. Then, in fact, Kamon & Ohki (2013) proved that the algorithm loses the purity of quantum states during the consensus operations. Purity is an important quantity for application to quantum information technology and above fact is not desirable for the purpose of generating useful quantum states. The similar approaches for consensus with no feedback control (e.g. Sepulchre et al. (2010); Shi et al. (2015, 2016)) also have this issue.

Motivated by the above fact, Kamon & Ohki (2013) and Kamon & Ohki (2014) have proposed a hybrid type of the distributed consensus algorithm and a distributed feedback with quantum state observation; projective measurements, in order to realize SSC and high purity simultaneously. They have shown the efficiency of their control scheme by numerical simulations and further expected that their algorithm realizes artificial bosonization or artificial fermionization.

The convergence of their algorithm, however, has not been proved and left for as an open problem. Then, in this paper, we tackle with this open problem and solve it completely. For more details, we modify the algorithm proposed by Kamon & Ohki (2013, 2014) and give a strict proof of the convergence to SSC from arbitrary initial states keeping purity. The difficulty of the proof is from that the dynamics is governed by two types of stochastic processes; (1) probabilistic selection of local subsystems among the whole networked quantum system, (2) feedback control action depending on the probabilistic quantum observation results, projective measurements, of the selected local subsystems. Therefore, the feedback control dynamics depends on the state-depending complicated combinations of the above stochastic processes and, as a result, it is represented as "a stochastic non-linear equation." In fact, a simple idea of applying the Kraus map discussed in Mazzarella et al. (2013) for their "deterministic linear autonomous systems" is not applicable in our case and its analysis requires strict dealing with the dynamics and the combinations as discussed in our paper.

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To overcome the complexity of stochastic non-linear systems, we employ the stochastic version of the Lyapunov stability theory, which is a well known method in quantum control Mirrahimi & van Handel (2007). Moreover, we also show that the proposed algorithm can generate a W-state, which cannot be obtained by the algorithm of Mazzarella *et al.* because the purity of the W-state is maximum among all the quantum states. It is well known that the W-state is one of the significant quantum entangled states and utilized in wide areas such as quantum memory. This is an important application of quantum consensus generation.

The similar research on the quantum consensus with feedback control is in Mazzarella et al. (2015), where the target state is restricted to an eigenstate and entangled states are not realized. Ticozzi (2016) recently reports the similar result of this paper in a deterministic way, however it is unclear whether its assumed deterministic operation can be realized by a feedback strategy which essentially depends on the probabilistic observation output by projective measurements.

This paper is organized as follows. In Section 2, we introduce some mathematical preliminaries and define the problem setting. In Section 3, we show the main results of this paper and prove them. In Section 4, we show numerical examples to confirm the efficiency of the proposed algorithm. Finally, we conclude this paper in Section 5.

Note that we omit many of the proofs for lemmas in this paper from the page limitation.

2. FORMULATION

2.1 Convergence of Stochastic System

Let $\{x_n\}_{n\in\{0\}\cup\mathbb{N}} \subset \mathbb{C}^m$ be a sequence of random variables. Then, we introduce definitions of convergence as follows:

Definition 1. A sequence $\{x_n\}$ is said to converge to $\tilde{x}$ in probability if $\lim_{n\to\infty} \mathbb{P}\{|x_n - \tilde{x}| \geq \epsilon\} = 0$ for any $\epsilon > 0$.

Definition 2. A sequence $\{x_n\}$ is said to converge to $\tilde{x}$ with probability one (w.p.1) if $\mathbb{P}\{\lim_{t\to\infty} x_n = \tilde{x}\} = 1$.

It is known that convergence w.p.1 is stronger than convergence in probability.

In this paper, we deal with a quantum control system which makes a quantum state converge to a target state in the above probabilistic sense. In order to show such convergence, we employ the following stochastic Lyapunov stability theorem.

Definition 3. A set $\mathcal{C}$ is called an invariant set if any initial state of a dynamical system belonging to $\mathcal{C}$ never leaves $\mathcal{C}$.

Proposition 1. (Kushner (1971)) Let $\{x_n\}_{n\in\{0\}\cup\mathbb{N}} \subset \mathbb{C}^m$ be a state of some dynamical system and a Markov process. Assume that there exist bounded non-negative functions $V(x)$ and $k(x)$ which satisfy

$$\mathbb{E}\{V(x_n)|x_{n-1}\} - V(x_{n-1}) = -k(x_{n-1}) \qquad (1)$$

for all $n \in \mathbb{N}$. Then, $k(x_n) \to 0$ $(n \to \infty)$ for almost all the paths. In addition, let $\mathcal{M} = \{x \in \mathbb{C}^m \mid k(x) = 0\}$, and let $\tilde{\mathcal{M}}$ be the largest invariant set of $\mathcal{M}$, then x_n converges to $\tilde{\mathcal{M}}$ in probability.

In some cases, convergence in probability implies convergence w.p.1.

2.2 Quantum State and Quantum Consensus State

In this paper, we deal with a multipartite quantum system composed of N isomorphic subsystems, labeled with indices $i = 1, 2, \ldots, N$, with associated Hilbert space $\mathcal{H}^N := \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_N$, with $\dim(\mathcal{H}_i) = D$ for all i and D is an integer satisfying $D \geq 2$. Let $\{|d_i\rangle\}_{d_i\in\{0,1,\ldots,D-1\}}$ be a set of basis vectors of $\mathcal{H}_i$, then the basis vectors of $\mathcal{H}^N$ are represented by $\{|d_1\rangle\otimes|d_2\rangle\otimes\cdots\otimes|d_N\rangle\}_{\forall i, d_i\in\{0,1,\ldots,D-1\}}$. Hereafter, we abbreviate $|d_1\rangle\otimes|d_2\rangle\otimes\cdots\otimes|d_N\rangle$ to $|d_1 d_2 \cdots d_N\rangle$. In addition, we regard $\mathcal{H}_i$ as $\mathbb{C}^D$ and the basis vector $|d_i\rangle$ as $(0\ 0\ \cdots\ 0\ 1\ 0\ 0\ \cdots\ 0)^\top$, i.e., the $d_i + 1$-th element is one and the others are zero, where $\top$ is a transpose operator.

Remark 1. Our main results are obtained in the case of $D = 2$, while some of lemmas are also true in the general case. So, we specify the condition $D = 2$ only if necessary in the following.

Define $\mathcal{B}(n)$ as a set of matrices with dimension $n \times n$. Then a quantum state on $\mathcal{H}^N$ is represented by a density matrix in

$$\mathfrak{D}(D^N) := \{\rho \in \mathcal{B}(D^N) | \ \rho = \rho^\dagger \succeq 0, \mathrm{tr}(\rho) = 1\}, \qquad (2)$$

where $\dagger$ is the complex conjugate transpose operator, $\succeq 0$ means that the matrix is semi-positive definite, and $\mathrm{tr}(\cdot)$ is an trace operator. A density matrix completely represents a probability distribution of a quantum system.

In particular, if the rank of a density matrix is one, the quantum state is called a pure state, while it is called a mixed state if the rank is larger than one. A pure state ρ is completely expressed by a state vector ψ as $\rho = \psi\psi^\dagger$, which is an element of $\mathfrak{D}'(D^N) := \{\psi \in \mathbb{C}^{D^N} \mid \|\psi\| = 1\}$, where $\|\cdot\|$ is 2-norm.

In this paper, we deal with a networked quantum system and consider to realize a quantum consensus state, called symmetric state consensus (SSC) introduced by Mazzarella et al. (2015) as follows:

Definition 4. (Mazzarella et al. (2015)) Let π be a permutation of integers $1, 2, \ldots, N$, and let $U_\pi \in \mathcal{B}(D^N)$ be a permutation matrix satisfying $U_\pi(x_1 \otimes x_2 \otimes \cdots \otimes x_N) = x_{\pi(1)} \otimes x_{\pi(2)} \otimes \cdots \otimes x_{\pi(N)}$ for all $\{x_n\}_{n=1}^N \subset \mathbb{C}^D$. Then, a quantum state $\rho \in \mathfrak{D}(D^N)$ is called in symmetric state consensus (SSC) if $U_\pi \rho U_\pi^\dagger = \rho$ holds with any π.

We also use the notation SSC to represent the set of all the quantum states in symmetric state consensus.

2.3 Network Structure and Quasi-local Operation

Mazzarella et al. (2015) introduced a consensus algorithm to realize SSC, however Kamon & Ohki (2013) proved that it loses purity

$$\mathrm{tr}(\rho^2). \qquad (3)$$

Motivated by this fact, in the following of this section, we introduce a networked quantum system composed of several quantum subsystems with a hybrid type (Kamon & Ohki (2013, 2014)) of a quantum consensus algorithm (Mazzarella et al. (2015)) and feedback control with observations, that is, distributed quasi-local measurements and quasi-local feedback controls, in order to realize SSC and high purity simultaneously.

The network structure is represented by a graph $\mathcal{G}_N = (\mathcal{V}_N, \mathcal{E}_N)$, where $\mathcal{V}_N$ is a node set and $\mathcal{E}_N$ is a branch set. A node corresponds to a subsystem and an edge means that there exists a set of measurement and feedback control which "locally" acts on the connected two subsystems in the quantum sense. On the network topology, we assume the following:

Assumption 1. (Kamon & Ohki (2013, 2014)) The graph $\mathcal{G}_N$ is connected.

It is known that when $\mathcal{G}_N$ is connected, there always exists a node $v_i \in \mathcal{V}_N$ s.t. a subgraph $\mathcal{G}_{N-1}$ of $\mathcal{G}_N$ obtained by removing v_i and the connected edges to v_i from $\mathcal{G}_N$ is still connected. Repeat this operation and we can get a sequence of subgraphs $\mathcal{G}_N, \mathcal{G}_{N-1}, \ldots, \mathcal{G}_2$.

Next, we introduce a quasi-local measurement operator $\sigma_{i,j}$ on the two subsystems composed of i-th subsystem and j-th subsystem (we use "quasi-local" for representing "quantum local operation" to distinguish from classical cases). In the finite dimensional quantum systems, a measurement operator, i.e., an observable is represented by a Hermitian matrix, and the measurement projects quantum states onto an eigenspace corresponding to the measured value, which is equal to the eigenvalue of the Hermitian matrix. Then, Kamon & Ohki (2013, 2014) defined $\sigma_{i,j}$ as below.

Definition 5. (Kamon & Ohki (2013, 2014))

$$\sigma_{i,j} := pP_{i,j} + qQ_{i,j} \quad (p \neq q \in \mathbb{R}), \tag{4}$$

where

$$P_{i,j} = \frac{1}{2}(I + S_{i,j}), \quad Q_{i,j} = \frac{1}{2}(I - S_{i,j}), \tag{5}$$

and $S_{i,j}$ is a swapping operator of subsystems i and j.

Remark 2. Equation (4) is a spectral decomposition of $\sigma_{i,j}$ with projection matrices $P_{i,j}$ and $Q_{i,j}$ and the corresponding eigenvalues p and q.

Remark 3. These measurements are quasi-local in a quantum sense and realizable in appropriate instruments.

Let us perform a measurement $\sigma_{i,j}$ on ρ, then we get a measured value p with a probability of $\mathrm{tr}(\rho P_{i,j})$ and the quantum state changes to ρ_p, or we get a measured value q with a probability of $\mathrm{tr}(\rho Q_{i,j})$ and the quantum state changes to ρ_q, where

$$\rho_p := \frac{P_{i,j}\rho P_{i,j}}{\mathrm{tr}(\rho P_{i,j})}, \quad \rho_q := \frac{Q_{i,j}\rho Q_{i,j}}{\mathrm{tr}(\rho Q_{i,j})}. \tag{6}$$

If ρ is a pure state, ρ_p and ρ_q are also pure states. Then, in that case, we represent $\rho = \psi\psi^\dagger$, and $\rho_p = \psi_p\psi_p^\dagger$ and $\rho_q = \psi_q\psi_q^\dagger$ where $\psi_p = P_{i,j}\psi/\|P_{i,j}\psi\|$ and $\psi_q = Q_{i,j}\psi/\|Q_{i,j}\psi\|$.

Note that ρ_p is symmetric about i and j, i.e., invariant under a permutation of subsystems i and j, while ρ_q is anti-symmetric about i and j. Since ρ_q is an undesirable state to attain SSC, we perform a feedback control with a unitary matrix $U_{i,j}$ on the subsystems i and j when the measured value is q to drive the quantum states from the eigenspace of $Q_{i,j}$ onto that of $P_{i,j}$. In this paper, we consider applying the following $U_{i,j}$ as a unitary operation.

Definition 6. Let $D = 2$. A unitary matrix $U_{i,j} = I_2 \otimes \cdots \otimes I_2 \otimes U \otimes I_2 \otimes \cdots \otimes I_2 \in \mathcal{B}(D^N)$ as a quasi-local operation acts on the subsystems i and j as $U = \mathrm{diag}(1, 1, -1, 1)$ and on the other subsystems as identity operator.

Since U is expressed by $\sigma_z \otimes I_2$, where σ_z is a Pauli matrix $\sigma_z = \mathrm{diag}(1, -1)$, $U_{i,j}$ is a phase reverse operation about z-axis in the case of spin systems. A quantum state ρ changes to $U_{i,j}\rho U_{i,j}^\dagger$ by the unitary operation $U_{i,j}$. If ρ is a pure state and $\rho = \psi\psi^\dagger$, ψ changes to $U_{i,j}\psi$.

2.4 Consensus Algorithm with Distributed Feedback Control

We introduce a new quantum consensus algorithm (QCA) with feedback in order to globally achieve SSC with high purity w.p.1. This is based on the algorithm proposed by Kamon & Ohki (2013, 2014).

(QCA)

(1) Randomly select an edge (i, j) from the branch set $\mathcal{E}_N$.
(2) Measure the quantum state by $\sigma_{i,j}$.
(3) Perform a unitary operation $U_{i,j}$ if the measured value is q, or do nothing otherwise. Then, back to (1).

There are two modifications from the original algorithm proposed by Kamon & Ohki (2013, 2014). One is that we select an edge (i, j) randomly, not by rotation. The other is that we specify the unitary operation $U_{i,j}$ in Definition 6 with which we succeed in guaranteeing the convergence to SSC, while Kamon & Ohki (2013, 2014) cannot find such unitary operation.

Remark 4. Note that the measurements of the projections do not decrease the purity and the unitary operation $U_{i,j}$ keeps it unchanged. Therefore, it is obvious that the purity does not decrease in QCA.

In this paper, we further show that QCA can be applied to generate a W-state, which is one of the important entangled states. A W-state is represented as $\rho^W = \psi^W\psi^{W\dagger} \in \mathfrak{D}(2^N)$, where

$$\psi^W = \frac{1}{\sqrt{N}}(|100\cdots0\rangle + |010\cdots0\rangle + \cdots + |00\cdots01\rangle). \tag{7}$$

It is obvious that ρ^W is in SSC. If one of the N-qubits is lost, the remaining $(N-1)$-qubits keep a W-state in a sense that $\rho_N^W = |0\rangle\langle 0| \otimes \rho_{N-1}^W + |1\rangle\langle 1| \otimes |00\cdots0\rangle\langle 00\cdots0|$, where ρ_N^W (ρ_{N-1}^W, ρ_{N-2}^W, ...) represents N-qubits ($(N-1)$-qubits, $(N-2)$-qubits, ...) W-state. This is one of the interesting properties of the W-state and its robustness is highly appreciated in the field of quantum memory.

3. MAIN RESULT

Our main results are the following Theorem 5 and Corollary 6.

Theorem 5. Let $D = 2$, then QCA drives quantum states into SSC w.p.1 from arbitrary initial states.

Corollary 6. Let the initial state be $\rho_{\mathrm{I}}^W = \psi_{\mathrm{I}}^W \psi_{\mathrm{I}}^{W\dagger}$, where $\psi_{\mathrm{I}}^W = |100\cdots 0\rangle$, then the quantum state converges to a W-state w.p.1 with QCA.

Remark 7. A significance of Theorem 5 is that there is no assumption about the structure of $\mathcal{G}_N$ except for Assumption 1, i.e., connectedness.

Remark 8. From Theorem 5, it is known that the convergent state by QCA in SSC depends on the initial state, and from Corollary 6, it converges to a W-state if its initial state is ρ_{I}^W. Note that ρ_{I}^W is relatively easy to prepare in spin systems rather than to prepare entangled states directly. This implies the efficiency of QCA in actual applications. Note that Corollary 6 is a direct result from Theorem 5, however it represents the significance from the application viewpoint.

Remark 9. The entanglement between N-qubits is strengthened by the repetition of quasi-local measurements and quasi-local feedback controls.

3.1 Proof of Theorem 5

To simplify the following discussion, we define a permutation matrix $T \in \mathcal{B}(D^N)$ in the following. At first, classify the bases of $\mathcal{H}^N$ into the equivalence classes so that for all the pairs of two bases in a equivalence class, there exists a permutation of the subsystems s.t. one of the pair changes to the other by the permutation. For example, when $D = 2$ and $N = 3$, the 2^3 bases are classified into four equivalence classes, $\{|000\rangle\}$, $\{|001\rangle, |010\rangle, |100\rangle\}$, $\{|011\rangle, |101\rangle, |110\rangle\}$, $\{|111\rangle\}$. Let $\mathcal{F}_1$, $\mathcal{F}_2$, $\ldots$, $\mathcal{F}_{k_{\max}}$ denote the equivalence classes, where $k_{\max}$ denotes the number of the equivalence classes and l_k denotes the number of the elements of $\mathcal{F}_k$, which satisfies $1 \le l_k, \sum_{k=1}^{k_{\max}} l_k = D^N$. Then, construct T as follows. First, arrange the elements of $\mathcal{F}_1$ from the first column to the l_1-th column in T. Next, arrange the elements of $\mathcal{F}_2$ from the l_1+1-th column to the l_2-th column in T. Repeating this procedure for all $\mathcal{F}_k$, we get a permutation matrix T. By the permutation of T, for example, the permutation matrix $S_{i,j}$, which represents the permutation between the i-th subsystem and the j-th subsystem, is represented as the following block-diagonal matrix;

$$T^\top S_{i,j} T = \begin{pmatrix} S_{i,j}(1) & & & & \\ & \ddots & & O & \\ & & S_{i,j}(k) & & \\ & O & & \ddots & \\ & & & & S_{i,j}(k_{\max}) \end{pmatrix}, \quad (8)$$

where O is a zero matrix and the size is represented in the subscript if necessary in the following, and $\forall k \in \{1, 2, \ldots, k_{\max}\}$, $S_{i,j}(k) \in \mathcal{B}(l_k)$. We abbreviate (8) to $\mathrm{blkdiag}(S_{ij}(1), \ldots, S_{ij}(k_{\max}))$ or $\mathrm{blkdiag}_{k=1}^{k_{\max}}(S_{ij}(k))$. We also use the same representation in the following discussion to describe block-diagonal matrices.

As mentioned in Section 1, the feedback control dynamics depends on the state-depending complicated combinations of two types of stochastic processes and it is represented as a stochastic non-linear equation. An idea of applying the Kraus map for linear autonomous systems is not applicable in this case and its analysis requires strict dealing with the dynamics and the combinations as follows.

In order to prove Theorem 5, first we constitute a bounded non-negative function $V(\cdot)$ which satisfies the assumption of Proposition 1, i.e., $V(\cdot)$ is a Lyapunov function, and then show that quantum states converge to SSC in probability with QCA. Finally, we show that they also converge to SSC w.p.1.

<u>(1) Construction of a Lyapunov function</u>

If $\rho \in$ SSC, ρ is invariant under any permutation of subsystems, and also invariant under any projection $P_{i,j}$. As $P_{i,j}$ is a projection matrix, $\rho \in$ SSC belongs to the common eigenspace of all $P_{i,j}$ whose corresponding eigenvalues are 1. Therefore, we can construct a projection matrix onto SSC as below.

Lemma 10. Consider

$$\left(\prod_{(i,j)\in\mathcal{E}_N} P_{i,j} \right)^n, \quad (9)$$

and (9) converges to the following $\tilde{P}_N$ as $n \to \infty$ regardless of the order of products;

$$\tilde{P}_N := T(\mathrm{blkdiag}_{k=1}^{k_{\max}}(\tilde{P}_N(k)))T^\top,$$
$$\tilde{P}_N(k) := \frac{1}{l_k}\mathrm{ones}(l_k, l_k), \quad (10)$$

where $\mathrm{ones}(n,n)$ is a matrix in $\mathcal{B}(n)$ whose elements are all 1.

We show some useful properties of $\tilde{P}_N$ hereafter. At first, we can also define $\tilde{P}_m$ as $\tilde{P}_m := \lim_{n\to\infty}(\prod_{(i,j)\in\mathcal{E}_m} P_{i,j})^n$ for $m \in \{2, 3, \ldots, N\}$ and get the following:

Lemma 11. Let $m \in \{2, 3, \ldots, N\}$, then the following hold for all $(i,j) \in \mathcal{E}_m$;

$$\tilde{P}_m P_{i,j} = P_{i,j}\tilde{P}_m = \tilde{P}_m,\ \tilde{P}_m Q_{i,j} = Q_{i,j}\tilde{P}_m = O,$$
$$\tilde{P}_m^2 = \tilde{P}_m, \quad (11)$$
$$\mathrm{tr}(\rho P_{i,j}) = 0 \ \Rightarrow\ \mathrm{tr}(\rho \tilde{P}_m) = 0. \quad (12)$$

Lemma 11 implies that $\tilde{P}_m$ is a projection matrix. Furthermore, we can show that $\tilde{P}_N$ is also a projection matrix on SSC as follows:

Lemma 12.

$$\mathrm{tr}(\rho\tilde{P}_N) = 1 \ \Rightarrow\ \rho \in \mathrm{SSC} \quad (13)$$

We now define a bounded non-negative function, that is, a Lyapunov function $V(\cdot)$ as

$$V(\rho) = 1 - \mathrm{tr}(\rho\tilde{P}_N). \quad (14)$$

It is known that $V(\rho)$ satisfies $0 \le V(\rho) \le 1$ due to $0 \le \mathrm{tr}(\rho\tilde{P}_N) \le 1$ and $V(\rho) = 0 \Rightarrow \rho \in$ SSC because of Lemma 12.

Then, we show that $V(\rho)$ satisfies the condition (1) in Proposition 1. Let $\{\rho_n\}_{n\in\{0\}\cup\mathbb{N}}$ be a sequence of quantum states, where ρ_0 is the initial state and ρ_n is generated by iterating QCA n times. Define the conditional expectation of the increment of $V(\rho_n)$ as

$$\Delta V(\rho_n) := \mathbb{E}\{V(\rho_{n+1})|\ \rho_n\} - V(\rho_n), \quad (15)$$

and we get the following:

Lemma 13. The following holds:

$$\Delta V(\rho_n) \le 0,\ \forall i, j \tag{16}$$

Moreover,

$$\Delta V(\rho_n) = 0,\ \forall i, j\ \Leftrightarrow\ \mathrm{tr}(U_{i,j} Q_{i,j} \rho_n Q_{i,j} U_{i,j}^\dagger \tilde{P}_N) = 0,\ \forall i, j. \tag{17}$$

(2) Convergence to SSC in probability

Let $\tilde{\mathcal{M}}$ be the largest invariant set that satisfies $\Delta V(\rho) = 0$, and let us classify $\mathfrak{D}(D^N)$ into the following three sets;

$$\begin{aligned} \mathcal{M}_1 &= \{\ \rho \in \mathfrak{D}(D^N) \mid \mathrm{tr}(\rho \tilde{P}_N) = 1\ \}, \\ \mathcal{M}_2 &= \{\ \rho \in \mathfrak{D}(D^N) \mid 0 < \mathrm{tr}(\rho \tilde{P}_N) < 1\ \}, \\ \mathcal{M}_3 &= \{\ \rho \in \mathfrak{D}(D^N) \mid \mathrm{tr}(\rho \tilde{P}_N) = 0\ \}. \end{aligned} \tag{18}$$

Note that $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ are mutually exclusive and $\mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 = \mathfrak{D}(D^N)$. It is known that Lemma 12 implies $\mathcal{M}_1 \subset \tilde{\mathcal{M}}$. Therefore if $\tilde{\mathcal{M}} \subset \mathcal{M}_1$ also holds, then $\tilde{\mathcal{M}} = \mathcal{M}_1$ and ρ_n converges to SSC in probability by Proposition 1. In order to show that, it is enough to check $\tilde{\mathcal{M}} \cap (\mathcal{M}_2 \cup \mathcal{M}_3) = \emptyset$ and we show this in the following.

First, we can show the following:

Lemma 14. The following holds:

$$\tilde{\mathcal{M}} \cap \mathcal{M}_2 = \emptyset \tag{19}$$

Next, we prove $\tilde{\mathcal{M}} \cap \mathcal{M}_3 = \emptyset$. At first, define a state vector version of $\mathcal{M}_3$ as

$$\mathcal{M}_3' = \{\ \psi \in \mathfrak{D}'(D^N) \mid \|\tilde{P}_N \psi\| = 0\ \}, \tag{20}$$

where $\mathfrak{D}'(D^N)$ denotes the set of state vectors of dimension D^N and also define a state vector version of $\tilde{\mathcal{M}}$ as $\tilde{\mathcal{M}}'$, then the following Lemma 15 holds, which implies that we can assume that the quantum state is a pure state for the proof of $\tilde{\mathcal{M}} \cap \mathcal{M}_3 = \emptyset$.

Lemma 15.

$$\tilde{\mathcal{M}} \cap \mathcal{M}_3 = \emptyset\ \Leftrightarrow\ \tilde{\mathcal{M}'} \cap \mathcal{M}_3' = \emptyset \tag{21}$$

Lemma 15 assures that all we have to prove is that there exists a path of a non-zero measure which leaves $\mathcal{M}_3'$ through a finite iteration of QCA when its initial state is in $\mathcal{M}_3'$, and we give its proof in Lemma 17. For it, we prepare some notations and a lemma. The node $\mathcal{V}_N \setminus \mathcal{V}_{N-1}$ is relabeled 'N' and one of the adjacent nodes of it is relabeled as '$N-1$'. Then, we can define $U_{N-1,N}$, $Q_{N-1,N}$, and also $\tilde{P}_{N-1}$ for $\mathcal{G}_{N-1} = (\mathcal{V}_{N-1}, \mathcal{E}_{N-1})$ as similar to (10). Then, we can get the following:

Lemma 16. Let $D = 2$ and let $\psi \in \mathfrak{D}'(D^N)$, then

$$\tilde{P}_{N-1}\psi = \psi\ \&\ \tilde{P}_N \psi = 0\ \Rightarrow\ \tilde{P}_N U_{N-1,N} Q_{N-1,N} \psi \neq 0. \tag{22}$$

Lemma 16 leads to the following Lemma 17.

Lemma 17. Let $D = 2$, and let $\{\psi_n\}_{n \in \{0\} \cup \mathbb{N}}$ be a sequence of pure states generated by QCA from the initial state $\psi_0 \in \mathfrak{D}'(D^N)$. Then, if $\tilde{P}_{N-1}\psi_0 \neq 0$ and $\tilde{P}_N \psi_0 = 0$, there exists a path of a non-zero measure which leaves $\mathcal{M}_3'$ through a finite iteration of QCA.

Lemmas 16 and 17 hold when N is replaced with $N-1$ by the same discussions. Moreover, $\tilde{P}_2 \psi_0 \neq 0$ is always realized by the feedback control. Therefore, by mathematical induction, a pure state belonging to $\mathcal{M}_3'$ leaves $\mathcal{M}_3'$ in a finite iteration of QCA, which implies $\tilde{\mathcal{M}'} \cap \mathcal{M}_3' = \emptyset$. Then, $\tilde{\mathcal{M}} \cap \mathcal{M}_3 = \emptyset$ is shown by Lemma 15.

From the above discussion, $\tilde{\mathcal{M}} = \mathcal{M}_1$ holds and applying Proposition 1, it is shown that the quantum states converge to SSC in probability with QCA from arbitrary initial states.

(3) Convergence to SSC w.p.1

We employ the way used in Mirrahimi & van Handel (2007) for the proof. Since ρ converges to SSC in probability,

$$\lim_{n \to \infty} \mathbb{P}\{V(\rho_n) > \epsilon\} = 0,\ \forall \epsilon > 0. \tag{23}$$

As $0 \le V(\rho_n) \le 1$, we have

$$\mathbb{E}\{V(\rho_n)\} \le \mathbb{P}\{V(\rho_n) > \epsilon\} + \epsilon(1 - \mathbb{P}\{V(\rho_n) > \epsilon\}). \tag{24}$$

Thus, $\limsup_{n \to \infty} \mathbb{E}\{V(\rho_n)\} \le \epsilon,\ \forall \epsilon > 0$ holds by applying (23), which implies $\lim_{n \to \infty} \mathbb{E}\{V(\rho_n)\} = 0$. Proposition 1 assures $\Delta V(\rho_n) \to 0$ for almost all the paths. As $V(\rho_n)$ is bounded and Lemma 14 holds, $V(\rho_n)$ converges for almost all the paths. Therefore, we get $\mathbb{E}\{\lim_{n \to \infty} V(\rho_n)\} = 0$ by the dominated convergence theorem (Lipster et al. (2001)), which implies that ρ_n converges to SSC w.p.1 as $n \to \infty$.

3.2 Outline of the proof of Corollary 6

Let $\mathfrak{D}_k(2^N)$ be the set of quantum pure states where the corresponding state vectors are in the linear space of $\mathcal{F}_k$. Note that $\psi_\mathrm{I}^W = |100 \cdots 0\rangle \in \mathcal{F}_2$. Then, we can show that the measurements and the unitary operations of QCA act on each $\mathfrak{D}_k(2^N)$, $k = 1, 2, \ldots, N$, independently. Thus, if $\rho_0 \in \mathfrak{D}_k(2^N)$, $\rho_n \in \mathfrak{D}_k(2^N)$ for any $n \in \mathbb{N}$. As $\mathrm{SSC} \cap \mathfrak{D}_2(2^N) = \{\rho^W\}$, we get Corollary 6 by applying Theorem 5 to the case $\rho_0 = \rho_\mathrm{I}^W$.

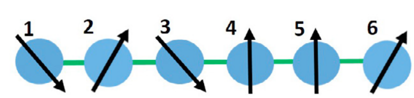


Fig. 1: An example of network structures of the quantum system. Each edge means that there exists a set of measurement and unitary operation between the connected two subsystems.

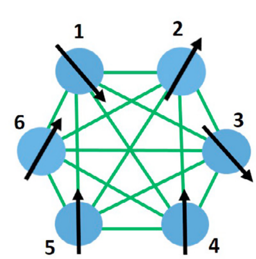


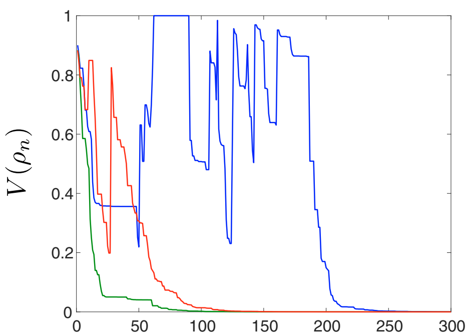
Fig. 2: Another example of network structures of the quantum system.

4. NUMERICAL EXAMPLE

In this section, we show some numerical examples.

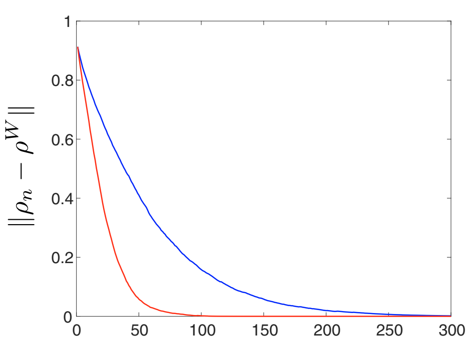
Fig. 3 shows the transitions of $V(\rho_n)$ by QCA with three random initial states, where the graph structure is given by Fig 1. Note that the outputs of the measurement of the system in QCA are probabilistic and we simulate it rigorously, then some of the plots show fluctuating transitions. In the green line, $V(\rho_n)$ monotonically decreases as n increases, while it is not the case with the red line and the blue line. Nevertheless, the quantum states converge to SSC in any case and we can confirm the efficiency of QCA.

Fig. 4 indicates the average of 1000 transitions of $\|\rho_n - \rho^W\|$ with the initial state ρ_{I}^W. Note that the blue line and the red line are the cases of Fig. 1 and Fig. 2, respectively. Fig. 4 supports the assertion of Corollary 6 and moreover suggests that the convergence rate gets higher as the number of edges increases, though the relation between the convergence rate and the graph structure needs further investigation.



n: the number of iterations of QCA

Fig. 3: Transitions of $V(\rho_n)$ by QCA with three random initial states. The graph structure is given by Fig 1.



n: the number of iterations of QCA

Fig. 4: Average of 1000 transitions of $\|\rho_n - \rho^W\|$ with the initial state ρ_{I}^W. The blue line and the red line are the cases of Fig. 1 and Fig. 2, respectively.

5. CONCLUSION

In this paper, we have shown that the quantum consensus algorithm (QCA) globally achieves SSC w.p.1 keeping purity by employing the stochastic version of the Lyapunov stability theory and it can be applied to generate a W-state, which is one of the important quantum states for quantum information technology.

There are some remaining issues. One is to research the relation between the graph structure of $\mathcal{G}_N$ and the convergence rate to SSC. As is well known, quantum states are very sensitive to noises and quantum entanglement collapses as time advances. Thus, it is desirable that the convergence rate is high. Second is to check if Theorem 5 is true when the choice of edges are done in a deterministic rotation as proposed in Kamon & Ohki (2013, 2014). Third is to give a proof in the case of $D \geq 3$ and enlarge the class of $U_{i,j}$, which increases the realizability of QCA. Final issue is the actual instrument for QCA.

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