NecSys'16, 8-9 September, 2016, Tokyo, Japan

Distributed Feedback Control of Quantum Networks

Reiji Takeuchi and Koji Tsumura†

THE UNIVERSITY OF TOKYO

The Aim of This Research

Systems: Networked quantum systems (e.g. connected spin systems) with local observation and local control.

Problem: Show an algorithm to attain

- symmetric state consensus (SSC) from arbitrary initial states with probability one and also to keep the purity,
- an important entangled state ($\in$ SSC), W-state.

Difficulty: A feedforward type quantum operation case [Mazzarella et al. (2013)] loses the purity [Kamon and Ohki (2013)].

Objective: Propose a distributed feedback control with local observation and local control and give the strict proof for the convergence to SSC or W-state with probability one.

Networked Quantum Systems

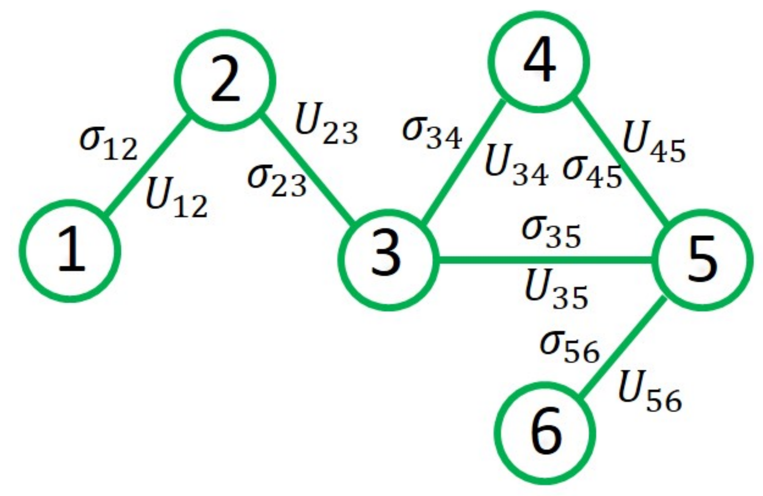


Fig. 1: An example of networked quantum systems with paired observations $\sigma_{i,j}$ and controls $U_{i,j}$ (the circles represent quantum subsystems)

- D: the dimension of each subsystem
- N: the number of subsystems $\Rightarrow$ the dimension of the whole quantum system is D^N
- $\mathcal{B}(n) := \{M \in \mathcal{C}^{n\times n}\}$
- $\mathfrak{D}(D^N) := \{M \in \mathcal{B}(D^N) | \, M = M^\dagger \succeq 0, \mathrm{tr}(M) = 1\}$
- $\rho \in \mathfrak{D}(D^N)$: quantum state
- $\mathcal{G}_N = (\mathcal{V}_N, \mathcal{E}_N)$: undirected graph
- $\mathcal{V}_N$: set of nodes,
- $\mathcal{E}_N$: set of undirected edges

- $(i, j) \in \mathcal{E}_N$: an undirected edge between node $i \in \mathcal{V}_N$ and node $j \in \mathcal{V}_N$
- $\mathcal{N}_i := \{j \in \mathcal{V}_N : (i, j)$ or $(j, i) \in \mathcal{E}_N\}$
- Assumption: $\mathcal{G}_N$ is connected.

Symmetric State Consensus (SSC) [Mazzarella et al. (2013)]:

- π: a permutation of integers $1, 2, \ldots, N$
- $U_\pi \in \mathcal{B}(D^N)$: a permutation matrix

$$U_\pi(x_1 \otimes x_2 \otimes \cdots \otimes x_N) = x_{\pi(1)} \otimes x_{\pi(2)} \otimes \cdots \otimes x_{\pi(N)}$$

- : symmetric state consensus (SSC) if $U_\pi \rho U_\pi^\dagger = \rho$ holds for any π

W-state:

$\rho^W = \psi^W \psi^{W\dagger} \in \mathfrak{D}(2^N)$ where

$$\psi^W := \frac{1}{\sqrt{N}}(|100\cdots 0\rangle + |010\cdots 0\rangle + \cdots + |00\cdots 01\rangle)$$

ρ^W is an entangled state and also an element of SSC

Lyapunov Function:

- $V(\rho) = 1 - \mathrm{tr}(\rho \tilde{P}_N)$
- $\tilde{P}_N$: projection on SSC
- $0 \le V(\rho) \le 1$
- $V(\rho) = 0 \Leftrightarrow \rho \in$SSC

Purity:

- purity $= \mathrm{tr}(\rho^2)$
- $0 \le \mathrm{tr}(\rho^2) \le 1$
- $\mathrm{tr}(\rho^2) = 1 \Leftrightarrow \rho$ is a pure state

Remark: We employ a stochastic Lyapunov theorem.

Consensus Algorithm with Distributed Feedback

Quantum Consensus Algorithm (QCA)

1. Randomly select an edge (i, j) from the edge set $\mathcal{E}_N$.
2. Measure the quantum state by $\sigma_{i,j}$, where
$$\sigma_{i,j} := pP_{i,j} + qQ_{i,j} \quad (p \neq q \in \mathcal{R}),$$
$$P_{i,j} := \frac{1}{2}(I + S_{i,j}), \quad Q_{i,j} := \frac{1}{2}(I - S_{i,j}),$$
and $S_{i,j}$ is a swapping operator of subsystems i and j.
3. Perform a unitary operation $U_{i,j}$ only to i and j such as $U_{i,j}\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}|01\rangle + |10\rangle$ (e.g. $U_{i,j} = \mathrm{diag}(1, -1) \otimes I_2$) if the measured value is q, or do nothing otherwise. Then, back to 1.

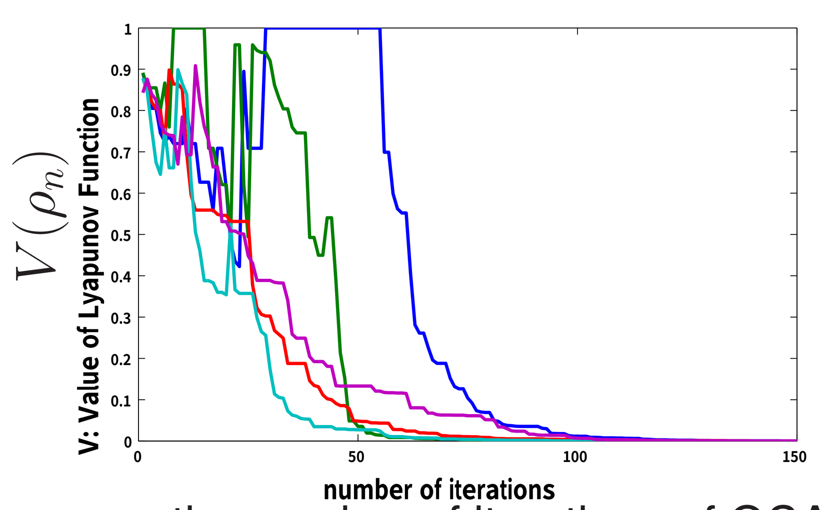
Remark: The measurements of the projections do not decrease the purity and the unitary operation $U_{i,j}$ keeps it unchanged. Therefore, the purity does not decrease in QCA.

Main Results

Theorem: Let $D = 2$, then QCA drives quantum states into SSC w.p.1 from arbitrary initial states.

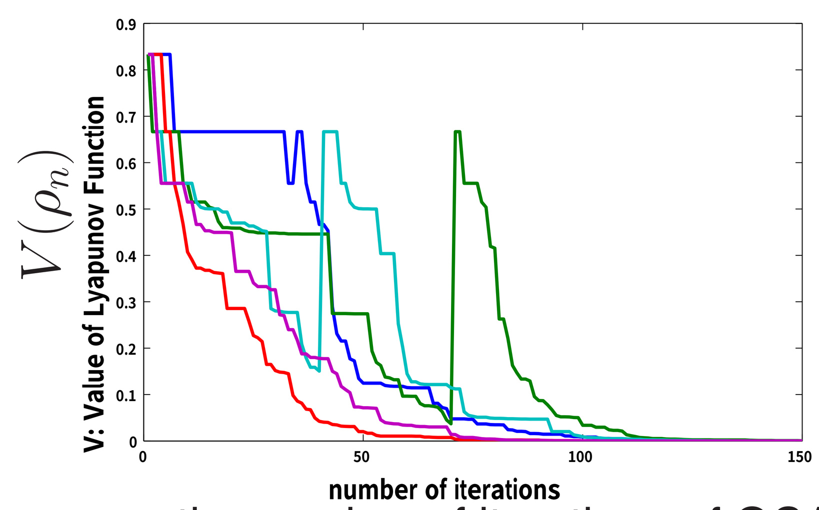
Corollary: Let the initial state be $\rho_1^W = \psi_1^W \psi_1^{W\dagger}$, where $\psi_1^W = |100\cdots 0\rangle$, then the quantum state converges to a W-state w.p.1 with QCA.

Numerical Simulations



n: the number of iterations of QCA

Fig. 2: Sample paths of Lyapunov function $V(\rho_n)$ of a system in Fig. 1 from five random initial states ρ_0.



n: the number of iterations of QCA

Fig. 3: Sample paths of Lyapunov function $V(\rho_n)$ of a system in Fig. 1 from ψ_1^W.

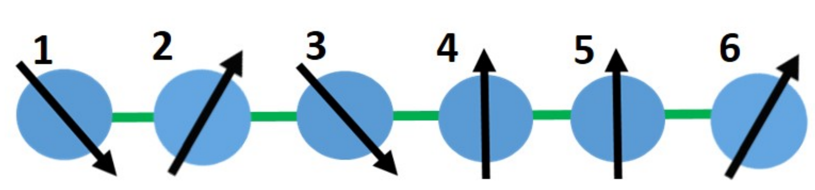


Fig. 4: Case of a cascade connection network structure.

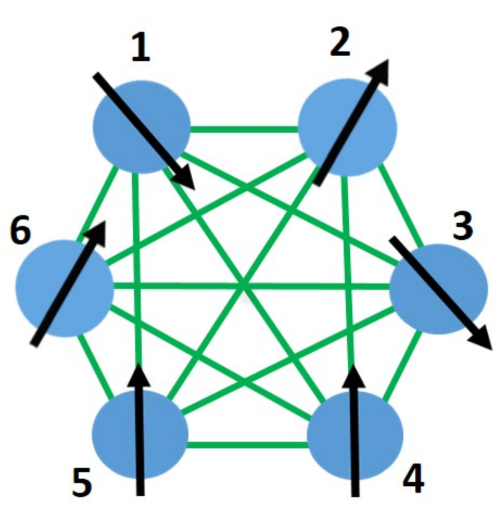
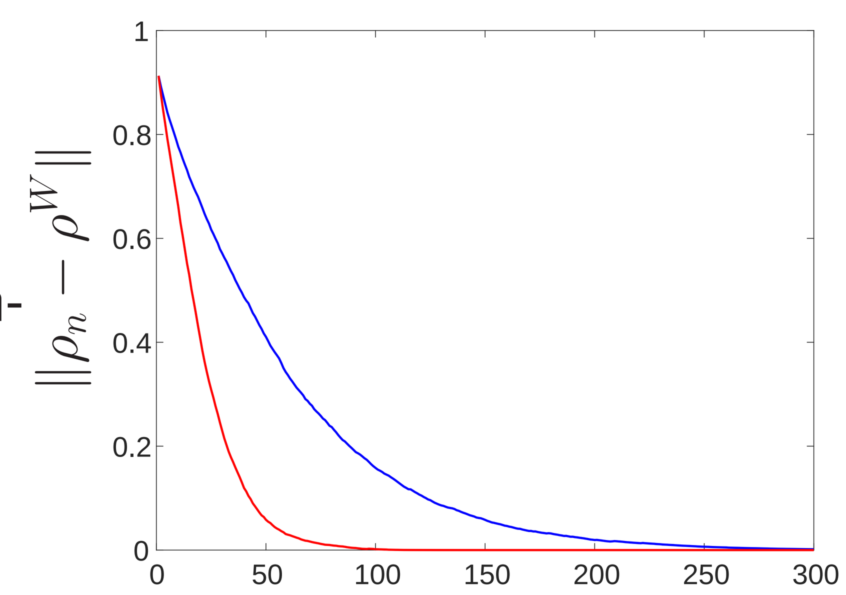


Fig. 5: Case of a strongly connected network structure.



n: the number of iterations of QCA

Fig. 6: Average of 1000 transitions of $\|\rho_n - \rho^W\|$ with the initial state ρ_1^W. The blue line and the red line are the cases of Fig. 4 and Fig. 5, respectively.

Kamon and Ohki (2013), Japan Joint Automatic Control Conference, vol. 56, 916, November 2013.

Mazzarella et al. (2013), arXiv, 1303.4077v1, quant-ph.

† tsumura@i.u-tokyo.ac.jp
http://www.cyb.ipc.i.u-tokyo.ac.jp/members/tsumu/index-e.html