**The Aim of This Research**

**Systems:** Networked quantum systems (e.g., connected spin systems) with local observation and local control.

**Problem:** Show an algorithm to attain
- symmetric state consensus (SSC) from arbitrary initial states with probability one and also to keep the purity,
- an important entangled state (∈ SSC), W-state.

**Difficulty:** A feedforward type quantum operation case [Mazzarella et al. (2013)] loses the purity [Kamon and Ohki (2013)].

**Objective:** Propose a distributed feedback control with local observation and local control and give the strict proof for the convergence to SSC or W-state with probability one.

**Networked Quantum Systems**

- $D$: the dimension of each subsystem
- $N$: the number of subsystems
- $E$: the set of directed edges
- $V$: the set of nodes
- $\mathcal{E}_N$: an undirected edge between node $i \in V_N$ and node $j \in V_N$
- $\mathcal{N}_i := \{j \in V_N : (i, j) \in \mathcal{E}_N\}$
- $\mathcal{N}_j := \{i \in V_N : (i, j) \in \mathcal{E}_N\}$
- $\mathcal{N}_i \cap \mathcal{N}_j = \emptyset$
- $\mathcal{E}_N \cup \mathcal{E}_N = \emptyset$
- $G_N = (V_N, \mathcal{E}_N)$: undirected graph
- $G_N$: set of nodes, $\mathcal{E}_N$: set of undirected edges
- $\sigma_{i,j}$: a permutation of integers

**Symmetric State Consensus (SSC) [Mazzarella et al. (2013)]:**

- $\pi$: a permutation of integers $1, 2, \ldots, N$
- $U_{\pi} \in \mathcal{B}(D^N)$: a permutation matrix

**W-state:**

$$\rho_W = \psi_W \psi_W^\dagger \in \mathcal{D}(2^N)$$

where

$$\psi_W := \frac{\sqrt{100\cdots0} + \sqrt{010\cdots0} + \sqrt{\cdots} + \sqrt{00\cdots01}}{\sqrt{2^N}}$$

**Lyapunov Function:**

$$V(\rho) = 1 - tr(\rho \rho_N)$$

$\rho_N$: projection on SSC

$$0 \leq V(\rho) \leq 1$$

$$V(\rho) = 0 \iff \rho \in \text{SSC}$$

**Purity:**

- $p^{\text{pure}} = tr(\rho^2)$
- $0 \leq p^{\text{pure}} \leq 1$
- $tr(\rho^2) = 1 \iff \rho$ is a pure state

**Consensus Algorithm with Distributed Feedback**

1. Randomly select an edge $(i, j)$ from the edge set $\mathcal{E}_N$.
2. Measure the quantum state by $\sigma_{i,j}$, where

$$\sigma_{i,j} := p_{i,j} \rho_{i,j} + q_{i,j} (p \neq q \in \mathbb{R})$$

$$p_{i,j} := \frac{1}{2}(I + S_{i,j})$$

$$q_{i,j} := \frac{1}{2}(I - S_{i,j})$$

and $S_{i,j}$ is a swapping operator of subsystems $i$ and $j$.
3. Perform a unitary operation $U_{i,j}$ only to $i$ and $j$ such as $U_{i,j} \frac{\sqrt{1}}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{\sqrt{1}}{\sqrt{2}}(|01\rangle + |10\rangle)$ (e.g., $U_{i,j} = \text{diag}(1, -1) \otimes I_2$) if the measured value is $\psi_W$, or do nothing otherwise. Then, back to 1.

**Remark:** The measurements of the projections do not decrease the purity and the unitary operation $U_{i,j}$ keeps it unchanged. Therefore, the purity does not decrease in QCA.

**Main Results**

**Theorem:** Let $D = 2$, then QCA drives quantum states into SSC w.p.1 from arbitrary initial states.

**Corollary:** Let the initial state be $\rho_W = \psi_W \psi_W^\dagger$, where $\psi_W = |100\cdots0\rangle$, then the quantum state converges to a W-state w.p.1 with QCA.

**Numerical Simulations**

- Fig. 2: Sample paths of Lyapunov function $V(\rho_i)$ of a system in Fig. 1 from five random initial states $\rho_0$.
- Fig. 3: Sample paths of Lyapunov function $V(\rho_i)$ of a system in Fig. 1 from five random initial states $\rho_0$.
- Fig. 4: Case of a cascade connection network structure.
- Fig. 5: Case of a strongly connected network structure.
- Fig. 6: Average of 1000 transitions of $\|\rho_W - \rho_W^\dagger\|$ with the initial state $\rho_W^\dagger$. The blue line and the red line are the cases of Fig. 4 and Fig. 5, respectively.

**Kamon and Ohki (2013), Japan Joint Automatic Control Conference, vol. 56, 916, November 2013.**

**Mazzarella et al. (2013), arXiv, 1303.4077v1, quant-ph.**

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http://www.cyb.ipc.i.u-tokyo.ac.jp/members/tsamura/index-e.html